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Functional integral approach of the two-dimensional massive fermion model with Thirring interaction among different species

L V Belvedere[1] **and A F Rodrigues**[2]

[1] Instituto de Física, Universidade Federal Fluminense, Av. Litorânea S/N, Boa Viagem, Niterói, CEP 24210-340, Rio de Janeiro, Brazil
[2] CBPF—Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, Urca, CEP 22290-180, Rio de Janeiro, Brazil

E-mail: belve@if.uff.br and aflavio@cbpf.br

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Abstract
We use the Abelian reduction of the Wess–Zumino–Witten theory to perform the functional integral bosonization of the two-dimensional fermion model with Thirring interaction among N different massive Fermi field species. The operator solution for the quantum equations of motion is reconstructed from the functional integral formulation. The fermion–boson correspondences, obtained earlier by Halpern, are generalized to the case of quartic interaction between different Fermi field species. For the massless model, the general Wightman functions of the Fermi field are displayed. The partition function and the equation of state of the statistical mechanical system associated with the effective bosonized theory are obtained. The charge screening mechanism for the Thirring field is discussed by considering the model with local gauge symmetry. The present approach provides a general guideline in order to obtain the operator solution of two-dimensional Abelian models, since we only need to know the free-field fermion–boson correspondences in order to reconstruct from the functional integral formulation the operator solution for the quantum equations of motion.

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1. Introduction

The bosonization of fermions, within the operator and functional integral approaches, has proven in the past to be a very useful technique for solving quantum field theoretical models in 1 + 1 dimensions [1].

In the efforts toward the extension of the bosonization procedure to 2 + 1 dimensions [2], and from the functional integral point of view, use has been made of an interpolating field procedure that leads to a mapping of the partition function of the original theory into a partition function of Chern–Simons-type theories. This same procedure has also been applied to discuss two-dimensional models [3–8]. However, the procedure of using auxiliary vector fields in order to decouple the Fermi fields, enlarges the field algebra and introduces redundant 'decoupled' Bose fields, in such a way that the Bose field algebra contains more degrees of freedom than those needed for the description of the physical content of the model. The structural aspects related to the appearance of decoupled massless Bose fields in the functional integral bosonization have not been fully appreciated and clarified in the literature. From our point of view, and within the functional integral approach, the equivalence between the two models should be established on the generating functional level, from which the Hilbert space of states is constructed.

In [9], the structural aspects of the functional integral bosonization of the massive Thirring model [10] have been analyzed. Using a synthesis of the functional integral and operator approaches and by considering the Abelian reduction of the Wess–Zumino–Witten theory (WZW) [11], Coleman's proof [12] of the fermion–boson mapping between the massive Thirring and the sine-Gordon theories is reconstructed in the Hilbert space of states.

The main aim of the present paper is to apply the synthesis of the functional integral and operator approaches presented in [9] to a generalized model involving the Thirring interaction among N different Fermi field species, in which we have a more complex Hilbert space structure than that of the standard Thirring model. This will enable the present work to fill a gap and clarify some structural aspects which remain obscure in the literature.

The functional integral bosonization of the two-dimensional fermion model with quartic interaction among different Fermi field species was discussed in [3, 4] on the partition function level. In [13] the particular case of the model with two fermion species is considered through the functional integral framework. Using the Abelian reduction of the Wess–Zumino–Witten theory the bosonized generating functional of the model is obtained and the fermion–boson mapping is established in the Hilbert space of states.

In this paper, we shall present a detailed study of the two-dimensional fermion model with quartic interaction among N different fermion species. The use of auxiliary vector fields in the bosonization procedure introduces redundant Bose fields. This leads to a structural problem concerning the construction of the true Hilbert space of states of the bosonized theory which is isomorphic to that of the original fermionic model. It is a common practice to discard these 'decoupled' massless scalar fields through the bosonization of the partition function [3, 4, 6]. From the mathematical point of view this is not a satisfactory procedure, and as stressed in [14–16], in some cases this procedure can lead to misleading conclusions on the physical content of the model. As we shall see, by performing the functional integral bosonization on the generating functional level, only zero norm combinations of these fields appear and they do not contribute to the Wightman functions of the Fermi field operator. Since these redundant field operators carry neither fermionic charge nor chirality, they reduce to the identity in the Hilbert space of states. This is a structural question which has not been fully appreciated and remains obscure in the literature on the functional integral bosonization of models in $1 + 1$ and $2 + 1$ dimensions.

From the operator point of view, a gap should be filled by providing the operator solution for the quantum equations of motion. Within the present approach, the operator solution is reconstructed from the functional integral formulation. The fermion–boson correspondences, early obtained by Halpern [17] for the case of the baryon-number current–current interaction, are generalized to the case of quartic interaction between different Fermi field species.

To emphasize the usefulness of the present functional integral approach for handling a large class of two-dimensional quantum field-theoretic models, we also consider the model with local $U(1)$ gauge symmetry and discuss the charge screening mechanism of the Thirring field. This streamlines the presentations of [18, 19] to the case of N flavored massive Fermi fields with quartic interaction among different fermion species.

To conclude the analysis of the model we also consider the statistical mechanical system associated with the effective bosonized Lagrangian. As is well known, Coleman's equivalence [12, 20] between the massive Thirring model and the sine-Gordon theory holds for the sine-Gordon parameter $\beta^2 < 8\pi$. Within the statistical mechanical point of view, this can be related to the fact that the equation of state of the associated statistical mechanical system of the sine-Gordon model [21] exhibits a Kosterlitz–Thouless (K–T) phase transition [22] at $\beta^2 = 8\pi$. In this paper we show how this phenomenon occurs in the massive fermion model with quartic interaction among N different fermion species, in which the partition function is given in terms of a 'multi-gas' expansion.

The paper is organized as follows: in section 2 we present general aspects of the model. In section 3, we use the Abelian reduction of the WZW theory [11] to obtain the generating functional in terms of the effective bosonized Lagrangian. In section 4, the operator solution for the quantum equations of motion is obtained and the Hilbert space structure is discussed. The fermion–boson correspondences, obtained earlier in Halpern's paper [17], are generalized to the case of quartic interaction between N different Fermi field species. In section 5 we discuss the model with local gauge symmetry, i.e., the QED_2 with N flavored massive Fermi fields with quartic interaction among different fermion species. The charge and chirality of the Thirring field are screened and carried by 'constant unitary operators' which generalize those obtained by Lowenstein and Swieca for the QED_2 [18]. The physical content of the model is described by $N-1$ sine-Gordon fields coupled to a massive sine-Gordon field with mass

$$m^2 = \frac{e^2 N}{\pi}\left(1 - \frac{g^2}{\pi}(N-1)\right)^{-1}.$$

In section 6, we consider the statistical mechanical system associated with the effective bosonized theory. The corresponding equation of state is obtained and exhibits a K–T phase transition at the critical dimension $D_c = 2$. The concluding remarks are presented in section 7. In appendix A the exact general Wightman functions of the model with massless fermions are presented. Some details of the computations of sections 5 and 6 are presented in appendices B and C. In appendix D, we discuss the problem of the regularization prescription in the controversial case with $N = 2$ and present our criticism with respect to the conclusions of [3].

2. General aspects

The model is defined by the classical Lagrangian[3],

$$\mathcal{L}(x) = \sum_{j=1}^{N} \bar{\psi}_j(x)(\mathrm{i}\gamma^\mu\partial_\mu - m_o)\psi_j(x) + g^2\sum_{j<k}^{N}(\bar{\psi}_j(x)\gamma^\mu\psi_j(x))(\bar{\psi}_k(x)\gamma_\mu\psi_k(x)), \tag{2.1}$$

[3] The conventions used are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1, \qquad \gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu,$$

$$g^{00} = -g^{11} = 1, \qquad \epsilon^{01} = 1$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial_\pm = \partial_0 \pm \partial_1 = \partial^\mp, \qquad (\partial_\mu\phi)^2 \equiv \partial_\mu\phi\partial^\mu\phi.$$

where j and k denote the fermion species. Defining the vector currents

$$\mathcal{J}_j^\mu = \bar{\psi}_j \gamma^\mu \psi_j, \tag{2.2}$$

the current–current interaction among different fermion species in the Lagrangian (2.1) can be written as

$$\mathcal{L}_I = \frac{1}{2} g^2 \left(\sum_{j=1}^{N} \mathcal{J}_j^\mu \right)^2 - \frac{1}{2} g^2 \sum_{j=i}^{N} \left(\mathcal{J}_j^\mu\right)^2, \tag{2.3}$$

where the $U(1)$ current $\mathcal{J}_\mu$ is defined by

$$\mathcal{J}^\mu = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathcal{J}_j^\mu. \tag{2.4}$$

The first term in the interaction Lagrangian (2.3) corresponds to a repulsive quartic-self-interaction for the $U(1)$ current and the second term corresponds to N independent attractive Thirring interactions for each species of currents $\mathcal{J}_j^\mu$. The classical equations of motion are given by

$$\mathrm{i}\gamma^\mu \partial_\mu \psi_j = -g^2 \left(\sum_{k=1}^{N} \mathcal{J}_k^\mu \right) \gamma_\mu \psi_j + g^2 \mathcal{J}_j^\mu \gamma_\mu \psi_j + m_o \psi_j. \tag{2.5}$$

In two dimensions the conserved vector current can be written in terms of the pseudo-scalar potential $\widetilde{\phi}_j$ as

$$\mathcal{J}_j^\mu = \epsilon^{\mu\nu} \partial_\nu \widetilde{\phi}_j. \tag{2.6}$$

Introducing the decomposition [1, 19]

$$\widetilde{\phi}_j = \frac{1}{\sqrt{N}} \widetilde{\phi} + \sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \widetilde{\phi}_{i_D}, \tag{2.7}$$

where λ^{i_D} are the $(N-1)$ mutually commuting generators of $SU(N)$ with normalization

$$\mathrm{tr}(\lambda^{i_D} \lambda^{j_D}) = \delta^{ij}, \tag{2.8}$$

$$\sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \lambda_{kk}^{i_D} = \left(\delta_{jk} - \frac{1}{N} \right), \tag{2.9}$$

the currents (2.6) can be written as

$$\mathcal{J}_j^\mu = \frac{1}{\sqrt{N}} \mathcal{J}^\mu + \sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \mathcal{J}_{i_D}^\mu, \tag{2.10}$$

and the classical equation of motion (2.5) is given by

$$\mathrm{i}\gamma^\mu \partial_\mu \psi_j = -g^2 \frac{(N-1)}{\sqrt{N}} \mathcal{J}^\mu \gamma_\mu \psi_j + g^2 \sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \mathcal{J}_{i_D}^\mu \gamma_\mu \psi_j + m_o \psi_j. \tag{2.11}$$

In the following sections, we shall use the functional integral bosonization in order to obtain the operator solution for the quantum version of the equation of motion (2.11).

For $N = 2$ the model exhibits a symmetry under the particle exchange $\psi_1 \leftrightarrow \psi_2$. In this case we can define the currents

$$\mathcal{J}_\pm^\mu = \tfrac{1}{\sqrt{2}} \left(\mathcal{J}_1^\mu \pm \mathcal{J}_2^\mu \right), \tag{2.12}$$

and the interaction Lagrangian (2.3) can be rewritten as

$$\mathcal{L}_I = \tfrac{1}{2}g^2\big(\mathcal{J}_+^\mu\big)^2 - \tfrac{1}{2}g^2\big(\mathcal{J}_-^\mu\big)^2. \tag{2.13}$$

The symmetry operation $\psi_1 \leftrightarrow \psi_2$ corresponds to $\mathcal{J}_\pm^\mu \leftrightarrow \pm\mathcal{J}_\pm^\mu$. As shown in [13] this symmetry is exhibited by the quantum theory in the physical range of the coupling constant for which $g^2 < \pi$.

The field algebra of the massless Thirring model ($m_o = 0$) is isomorphic to the algebra of the massless scalar field. For massless Fermi fields the model described by the Lagrangian (2.1) is a scale invariant theory with anomalous scale dimension. As in the standard Thirring model, in order for the theory described by the Lagrangian (2.1) to have the model with massless fermions as the short distance fixed point, the scale dimension of the mass operator must be [20]

$$D < 2. \tag{2.14}$$

In what follows the mass term should be understood as a perturbation in the scale invariant model.

3. Functional integral bosonization

In order to perform the functional integral bosonization of the model, we shall consider the generating functional and the functional integral identities

$$\mathrm{e}^{\mathrm{i}\int \mathrm{d}^2x\{\frac{1}{2}g^2(\sum_{i=1}^N \mathcal{J}_i^\mu)^2\}} \equiv \int \mathcal{D}a_\mu\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^2x\{-\frac{1}{2}g^2(a_\mu)^2 + g^2 a_\mu(\sum_{j=1}^N \mathcal{J}_j^\mu)\}}, \tag{3.1}$$

$$\mathrm{e}^{\mathrm{i}\int \mathrm{d}^2x\{-\frac{1}{2}g^2\sum_{j=1}^N(\mathcal{J}_j^\mu)^2\}} \equiv \int \prod_{j=1}^N \mathcal{D}b_j^\mu\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^2x\{\frac{1}{2}g^2\sum_{j=1}^N(b_{\mu_j})^2 + g^2\sum_{j=1}^N b_j^\mu \mathcal{J}_{\mu j}\}}. \tag{3.2}$$

This procedure enlarges the original field algebra by the introduction of two auxiliary vector fields (a_μ, b_μ) and the generating functional is given in terms of the functional integral measure

$$\prod_{j=1}^N \mathcal{D}\bar\psi_j \mathcal{D}\psi_j \to \left(\prod_{j=1}^N \mathcal{D}\bar\psi_j\mathcal{D}\psi_j\right)\mathcal{D}a_\mu \left(\prod_{j=1}^N \mathcal{D}b_{\mu_j}\right). \tag{3.3}$$

The effective Lagrangian is given by

$$\mathcal{L}_{\mathrm{eff}} = \sum_{j=1}^N \bar\psi_j(\mathrm{i}\gamma^\mu\partial_\mu - m_o)\psi_j - \frac{g^2}{2}(a_\mu)^2 + g^2 a_\mu\left(\sum_{j=1}^N \mathcal{J}_j^\mu\right) + \frac{g^2}{2}\sum_{j=1}^N\big(b_j^\mu\big)^2 + g^2\sum_{j=1}^N b_j^\mu \mathcal{J}_{\mu j}. \tag{3.4}$$

Defining N new vector fields

$$\mathcal{B}_j^\mu = b_j^\mu + a^\mu, \tag{3.5}$$

we obtain from (3.4)

$$\mathcal{L}_{\mathrm{eff}} = \sum_{j=1}^N \bar\psi_j(\mathrm{i}\gamma^\mu\partial_\mu + g^2\gamma^\mu\mathcal{B}_{\mu j} - m_o)\psi_j + \frac{g^2}{2}\sum_{j=1}^N\big(\mathcal{B}_j^\mu\big)^2 + \frac{g^2}{2}(N-1)(a_\mu)^2 - g^2 a_\mu\left(\sum_{j=1}^N \mathcal{B}_j^\mu\right). \tag{3.6}$$

Performing the functional integration over the field a_μ, the effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = \sum_{j=1}^{N} \bar{\psi}_j (\mathrm{i}\gamma^\mu \partial_\mu + g^2 \gamma^\mu \mathcal{B}_{\mu j} - m_o)\psi_j + \frac{g^2}{2}\sum_{j=1}^{N} \left(\mathcal{B}_j^\mu\right)^2 - \frac{1}{2}\frac{g^2}{(N-1)}\left(\sum_{j=1}^{N} \mathcal{B}_j^\mu\right)^2. \tag{3.7}$$

Introducing the vector field combinations

$$\mathcal{B}_\pm^j = \mathcal{B}_0^j \pm \mathcal{B}_1^j, \tag{3.8}$$

the fermionic piece of the Lagrangian can be written in terms of the two spinor components $\psi_{(\alpha)j}$ ($\alpha = 1, 2$) as follows ($\partial_\pm = \partial_0 \pm \partial_1$):

$$\mathcal{L}_F = \sum_{j=1}^{N} \left\{\psi_{(1)j}^\dagger \left(\mathrm{i}\partial_- + g^2 \mathcal{B}_-^j\right)\psi_{(1)j} + \psi_{(2)j}^\dagger \left(\mathrm{i}\partial_+ + g^2 \mathcal{B}_+^j\right)\psi_{(2)j} - m_o\left(\psi_{(1)j}^\dagger \psi_{(2)j} + \psi_{(2)j}^\dagger\right)\psi_{(1)j}\right\}. \tag{3.9}$$

In order to decouple the Fermi fields and the auxiliary vector fields in the Lagrangian (3.9), we introduce the parametrization of the fields $\mathcal{B}_\pm^j$ in terms of the $U(1)$ group-valued variables (U_j, V_j) as

$$\mathcal{B}_+^j = \frac{1}{g^2} U_j^{-1} \mathrm{i}\partial_+ U_j, \qquad \mathcal{B}_-^j = \frac{1}{g^2} V_j \mathrm{i}\partial_- V_j^{-1}, \tag{3.10}$$

such that

$$\begin{aligned}\psi_{(1)j}^\dagger &\left(\mathrm{i}\partial_- + g^2 \mathcal{B}_-^j\right)\psi_{(1)j} + \psi_{(2)j}^\dagger \left(\mathrm{i}\partial_+ + g^2 \mathcal{B}_+^j\right)\psi_{(2)j} \\ &= \left(\psi_{(1)j} V_j^{-1}\right)^\dagger \mathrm{i}\partial_-\left(V_j^{-1}\psi_{(1)j}\right) + (\psi_{(2)j} U_j)^\dagger \mathrm{i}\partial_+(U_j \psi_{(2)j}).\end{aligned} \tag{3.11}$$

The decoupling is achieved by performing the chiral rotations

$$\psi_{(1)j} = V_j \psi_{(1)j}^{(0)}, \qquad \psi_{(2)j} = U_j^{-1} \psi_{(2)j}^{(0)}, \tag{3.12}$$

where $\psi_j^{(0)}$ is the solution of the free massive Fermi theory with N species. Defining the covariant derivatives

$$D_\pm(\mathcal{B}^j) = \mathrm{i}\partial_\pm + g^2 \mathcal{B}_\pm^j, \tag{3.13}$$

and introducing in the functional integral the $2N$ identities

$$1 = \int \mathcal{D}U_j [\det D_+(U_j)]\delta\left(g^2 \mathcal{B}_+^j - U_j^{-1} \mathrm{i}\partial_+ U_j\right), \tag{3.14}$$

$$1 = \int \mathcal{D}V_j [\det D_-(V_j)]\delta\left(g^2 \mathcal{B}_-^j - V_j \mathrm{i}\partial_- V_j^{-1}\right), \tag{3.15}$$

the change of variables from $\left(\mathcal{B}_+^j, \mathcal{B}_-^j\right)$ to (U_j, V_j) is performed by integration over the fields $\mathcal{B}_\pm^j$. Performing the fermion chiral rotations (3.12) and taking into account the corresponding change in the integration measure [9, 23], we obtain

$$\left(\prod_j \mathcal{D}\bar{\psi}_j \mathcal{D}\psi_j\right)\left(\prod_j \mathcal{D}\mathcal{B}_\pm^j\right) = \left(\prod_j \mathcal{D}\bar{\psi}_j^{(0)} \mathcal{D}\psi_j^{(0)}\right)\left(\prod_j \mathcal{D}U_j \mathcal{D}V_j\right)\prod_j \mathcal{W}_j[U_j, V_j], \tag{3.16}$$

with

$$\mathcal{W}_j[U_j, V_j] = \mathrm{e}^{-\mathrm{i}\{\Gamma[U_j] + \Gamma[V_j] - \boldsymbol{b}\int \mathrm{d}^2x\, \mathcal{B}_+^j \mathcal{B}_-^j\}}, \tag{3.17}$$

where $\boldsymbol{b}$ is a regularization parameter and $\Gamma[G_j]$ is the Wess–Zumino–Witten (WZW) functional [9, 11], which enters in (3.17) with a negative level. In the Abelian case the WZW functional is given by

$$\Gamma[G] = \Gamma[G^{-1}] = \frac{1}{8\pi}\int \mathrm{d}^2 z\, \partial_\mu G^{-1}\partial^\mu G. \tag{3.18}$$

Due to the absence of local gauge invariance in the total effective theory, the last term in (3.17) has been added by exploiting the regularization freedom in the computation of the Jacobians. Since the effective fermionic Lagrangian is invariant under local gauge transformations we shall use the 'gauge invariant' regularization by setting[4]

$$\boldsymbol{b} = \frac{g^2}{4\pi}. \tag{3.19}$$

Using the Polyakov–Wiegman identity [24]

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\int \mathrm{d}^2 x (U^{-1}\partial_+ U)(V\partial_- V^{-1}), \tag{3.20}$$

we obtain[5]

$$\mathcal{W}_j = \mathrm{e}^{-\mathrm{i}\Gamma[U_j V_j]}. \tag{3.21}$$

The vector fields in two dimensions can be decomposed as

$$\mathcal{B}^\mu_j = \frac{1}{g^2}(\epsilon^{\mu\nu}\partial_\nu\widetilde{\phi}_j + \partial^\mu\zeta_j), \tag{3.22}$$

which corresponds to parametrizing the variables (U_j, V_j) as follows:

$$U_j = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_j+\zeta_j)}, \qquad V_j = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_j-\zeta_j)}, \tag{3.23}$$

and the Fermi fields are given by

$$\psi_j(x) = \mathrm{e}^{\mathrm{i}(\gamma^5\widetilde{\phi}_j(x)+\zeta_j(x))}\psi^{(0)}_j(x). \tag{3.24}$$

Taking all this into account, the total effective Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = \sum_j^N \Big\{ &\bar{\psi}^{(0)}_j \mathrm{i}\gamma^\mu\partial_\mu\psi^{(0)}_j - m_o\big(\psi^{(0)\dagger}_{(1)j}\psi^{(0)}_{(2)j}\,\mathrm{e}^{-2\mathrm{i}\widetilde{\phi}_j} + \psi^{(0)\dagger}_{(2)j}\psi^{(0)}_{(1)j}\,\mathrm{e}^{2\mathrm{i}\widetilde{\phi}_j}\big)\\ &- \frac{1}{2g^2}\left(1+\frac{g^2}{\pi}\right)(\partial_\mu\widetilde{\phi}_j)^2 + \frac{1}{2g^2}(\partial_\mu\zeta_j)^2\Big\}\\ &- \frac{1}{2}\frac{1}{g^2(N-1)}\left(\sum_{j=1}^N(\epsilon^{\mu\nu}\partial_\nu\widetilde{\phi}_j + \partial^\mu\zeta_j)\right)^2.\end{aligned} \tag{3.25}$$

Let us introduce the free-field bosonization expressions [1]

$$\psi^{(0)}_j(x) = \left(\frac{\mu_o^{2d}}{2\pi}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi\gamma^5} :\mathrm{e}^{\mathrm{i}\sqrt{\pi}\{\gamma^5\widetilde{\varphi}_j(x)+\int_{x^1}^\infty\partial_0\widetilde{\varphi}_j(x^0,z^1)\mathrm{d}z^1\}}:, \tag{3.26}$$

$$\bar{\psi}^{(0)}_j\gamma^\mu\partial_\mu\psi^{(0)}_j = \frac{1}{2}:(\partial_\mu\widetilde{\varphi}_j)^2:, \tag{3.27}$$

[4] A different choice of regularization implies a redefinition of the β parameter of the sine-Gordon theory and the physical range for the coupling constant g [9, 13].

[5] The Jacobian (3.21) is invariant under the Abelian 'gauge transformation' ${}^{\mathsf{g}}U = \mathsf{g}U, {}^{\mathsf{g}}V = V\mathsf{g}^{-1}$.

$$\psi^{(0)\dagger}_{(1)j}\psi^{(0)}_{(2)j} = \frac{\mu_o}{2\pi} : \mathrm{e}^{-2\mathrm{i}\sqrt{\pi}\widetilde{\varphi}_j} :, \tag{3.28}$$

where $:(\bullet):$ indicates normal ordering with respect to the free propagator $\left(\Box + \mu_o^2\right)^{-1}$ in the limit $\mu_o \to 0$ and d is the scale dimension of the Fermi field operator (the canonical dimension is $d = 1/2$).[6] Introducing the fields

$$\widetilde{\phi} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\widetilde{\phi}_j, \qquad \zeta = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\zeta_j, \tag{3.29}$$

the generating functional is given in terms of the bosonized effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \sum_{j=1}^{N}\left\{\frac{1}{2}(\partial_\mu\widetilde{\varphi}_j)^2 - \frac{1}{2g^2}\left(1 + \frac{g^2}{\pi}\right)(\partial_\mu\widetilde{\phi}_j)^2 + \frac{1}{2g^2}(\partial_\mu\zeta_j)^2 - m_o'\cos(2\sqrt{\pi}\widetilde{\varphi}_j + 2\widetilde{\phi}_j)\right\}$$
$$+\frac{1}{2}\frac{N}{g^2(N-1)}(\partial_\mu\widetilde{\phi})^2 - \frac{1}{2}\frac{N}{g^2(N-1)}(\partial_\mu\zeta)^2, \tag{3.30}$$

where $m_o' = \mu_o m_o/\pi$. Using the decomposition (2.7) and the same for the fields $\widetilde{\varphi}_j$ and ζ_j,

$$\widetilde{\varphi}_j = \frac{1}{\sqrt{N}}\widetilde{\varphi} + \sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\widetilde{\varphi}_{i_D}, \tag{3.31}$$

$$\zeta_j = \frac{1}{\sqrt{N}}\zeta + \sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\zeta_{i_D}, \tag{3.32}$$

the total bosonized effective Lagrangian can be rewritten as

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\varphi}_{i_D}\right)^2 + \frac{1}{2g^2(N-1)}\left(1 - \frac{g^2(N-1)}{\pi}\right)(\partial_\mu\widetilde{\phi})^2$$
$$-\frac{1}{2g^2}\left(1 + \frac{g^2}{\pi}\right)\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\phi}_{i_D}\right)^2 - \frac{1}{2}\frac{1}{g^2(N-1)}(\partial_\mu\zeta)^2 + \frac{1}{2g^2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\zeta_{i_D}\right)^2$$
$$-m_o'\sum_{j=1}^{N}\cos\left\{2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + \frac{2}{\sqrt{N}}\widetilde{\phi} + \sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}(2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\widetilde{\phi}_{i_D})\right\}. \tag{3.33}$$

Note that the range for the values of the coupling constant g^2 in (3.33) determines the sign for the metric of the field $\widetilde{\phi}$. As in the standard Thirring model, we shall consider

$$0 \leqslant g^2 < \pi/(N-1). \tag{3.34}$$

In order to have canonical fields we perform the field scaling

$$\widetilde{\phi} = \frac{g\sqrt{(N-1)}}{\sqrt{1 - \frac{g^2(N-1)}{\pi}}}\widetilde{\phi}', \tag{3.35}$$

$$\widetilde{\phi}_{i_D} = \frac{g}{\sqrt{1 + \frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}, \tag{3.36}$$

$$\zeta = g\sqrt{(N-1)}\zeta', \tag{3.37}$$

[6] For a detailed discussion of the meaning of $:(\bullet):$ in an interacting theory we refer the reader to [1, 20].

$$\zeta_{i_D} = g\zeta'_{i_D}, \tag{3.38}$$

and the Lagrangian (3.33) can be written as

$$\mathcal{L}_{\rm eff} = \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\varphi}_{i_D}\right)^2 + \frac{1}{2}(\partial_\mu\widetilde{\phi}')^2 - \frac{1}{2}\sum_{i_D=1}^{N-1}(\partial_\mu\widetilde{\phi}'_{i_D})^2 - \frac{1}{2}(\partial_\mu\zeta')^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\zeta'_{i_D}\right)^2$$
$$- m'_o\sum_{j=1}^{N}\cos\left\{2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + \frac{2}{\sqrt{N}}\frac{g\sqrt{(N-1)}}{\sqrt{1-\frac{g^2(N-1)}{\pi}}}\widetilde{\phi}' + \sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\left(2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}\right)\right\}. \tag{3.39}$$

Note that the fields $\left(\widetilde{\phi}'_{i_D}, \zeta'\right)$ are quantized with negative metric. The fields $(\widetilde{\varphi}, \widetilde{\phi}')$ act as pseudo-potentials for the $U(1)$ current $\mathcal{J}_\mu$ and the fields $\left(\widetilde{\varphi}_{i_D}, \widetilde{\phi}'_{i_D}\right)$ act as pseudo-potentials for the current $\mathcal{J}^\mu_{i_D}$. Taking into account the combination between the fields $(\widetilde{\varphi}, \widetilde{\phi}')$, as well as the combination between the fields $(\widetilde{\varphi}_{i_D}, \widetilde{\phi}'_{i_D})$ appearing in the mass term in (3.39), following the procedure introduced in [9, 13] we shall perform canonical field transformations. For the fields $(\widetilde{\varphi}, \widetilde{\phi}')$, both quantized with positive metric, we introduce the canonical transformation

$$\beta\widetilde{\Phi} = 2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + \frac{2}{\sqrt{N}}\frac{g\sqrt{(N-1)}}{\sqrt{1-\frac{g^2(N-1)}{\pi}}}\widetilde{\phi}', \tag{3.40}$$

$$\beta\widetilde{\xi} = \frac{2}{\sqrt{N}}\frac{g\sqrt{(N-1)}}{\sqrt{1-\frac{g^2(N-1)}{\pi}}}\widetilde{\varphi} - 2\sqrt{\frac{\pi}{N}}\widetilde{\phi}', \tag{3.41}$$

with

$$\beta^2 = \frac{4\pi}{N}\frac{1}{1-\frac{g^2}{\pi}(N-1)}. \tag{3.42}$$

Both fields $\widetilde{\Phi}$ and $\widetilde{\xi}$ have positive metric. For the fields $\left(\widetilde{\varphi}_{i_D}, \widetilde{\phi}'_{i_D}\right)$, which have opposite metric, we introduce the transformation

$$\gamma\widetilde{\Phi}_{i_D} = 2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}, \tag{3.43}$$

$$\gamma\widetilde{\xi}_{i_D} = 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\varphi}_{i_D} + 2\sqrt{\pi}\widetilde{\phi}'_{i_D}, \tag{3.44}$$

with

$$\gamma^2 = \frac{4\pi}{1+\frac{g^2}{\pi}}, \tag{3.45}$$

and the fields $\widetilde{\Phi}_{i_D}$ and $\widetilde{\xi}_{i_D}$ have opposite metric. The bosonized effective Lagrangian (3.39) is then given by

$$\mathcal{L}_{\rm eff} = \delta\mathcal{L}_o + \mathcal{L}\left[\widetilde{\Phi}, \widetilde{\Phi}_{i_D}\right], \tag{3.46}$$

where $\delta\mathcal{L}_o$ is the Lagrangian piece of the free massless scalar fields with opposite metric quantization,

$$\delta\mathcal{L}_o = -\frac{1}{2}(\partial_\mu\zeta')^2 + \frac{1}{2}(\partial_\mu\widetilde{\xi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\zeta_{i_D}\right)^2 - \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\xi}_{i_D}\right)^2, \tag{3.47}$$

and

$$\mathcal{L}[\widetilde{\Phi}, \widetilde{\Phi}_{i_D}] = \frac{1}{2}(\partial_\mu \widetilde{\Phi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1} \left(\partial_\mu \widetilde{\Phi}_{i_D}\right)^2 - m_o' \sum_{j=1}^{N} \cos\left(\beta\widetilde{\Phi} + \gamma \sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \widetilde{\Phi}_{i_D}\right). \tag{3.48}$$

The scale dimension of the mass operator in (3.48) is given by

$$D = \frac{1}{4\pi}\left(\beta^2 + \gamma^2 \frac{(N-1)}{N}\right). \tag{3.49}$$

Note that the N fields $(\widetilde{\Phi}, \widetilde{\Phi}_{i_D})$, which describe the physical content of the bosonized effective theory, have positive metric and the unitarity is not spoiled. It should be stressed that although the free and massless fields ζ', $\widetilde{\xi}$, ζ_{i_D} and $\widetilde{\xi}_{i_D}$ decouple in the Lagrangian, they do not decouple in the generating functional. As a matter of fact, these fields quantized with opposite metric are redundant bosonic degrees of freedom introduced by the use of auxiliary vector fields [9]. As we shall see, in the generating functional only zero norm combinations of these fields appear and they do not contribute to the Wightman functions of the Fermi field operator. As shown in [9], the factorization of the partition function will generally lead to incorrect conclusions concerning the physical content of the model.

For $g = 0$, $\beta^2 = 4\pi/N$, $\gamma^2 = 4\pi$, defining N bosonic fields

$$\widetilde{\Phi}_j \doteq \frac{1}{\sqrt{N}}\widetilde{\Phi} + \sum_{i_D=1}^{N-1} \lambda_{jj}^{i_D} \widetilde{\Phi}_{i_D}, \tag{3.50}$$

the Lagrangian (3.48) can be rewritten as

$$\mathcal{L}^{(0)} = \frac{1}{2}\sum_{j=1}^{N} (\partial_\mu \widetilde{\Phi}_j)^2 - m_o' \sum_{j=1}^{N} \cos(2\sqrt{\pi}\widetilde{\Phi}_j), \tag{3.51}$$

which corresponds to the bosonized Lagrangian of N free massive Fermi fields.

The bosonized Fermi-mass term in the Lagrangian (3.48) is the generalization to the case of interaction among different fermion species of the bosonized mass operator obtained by Halpern [17] for the case of the 'baryon-number current–current interaction' $g^2\left(\sum_{j=1}^{N} \mathcal{J}_j^\mu\right)^2$.

For $N = 2$, defining the fields $\widetilde{\Phi}_+ \equiv \widetilde{\Phi}$ and $\widetilde{\Phi}_- \equiv \widetilde{\Phi}_{i_D}$, the Lagrangian (3.48) can be rewritten as

$$\mathcal{L}[\widetilde{\Phi}_+, \widetilde{\Phi}_-] = \tfrac{1}{2}(\partial_\mu \widetilde{\Phi}_+)^2 + \tfrac{1}{2}(\partial_\mu \widetilde{\Phi}_-)^2 - m_o' \cos(\beta_+ \widetilde{\Phi}_+)\cos(\beta_- \widetilde{\Phi}_-), \tag{3.52}$$

where

$$\beta_+^2 = \frac{2\pi}{1 - \frac{g^2}{\pi}}, \qquad \beta_-^2 = \frac{2\pi}{1 + \frac{g^2}{\pi}}. \tag{3.53}$$

In the bosonized theory, the symmetry under the particle exchange $\psi_1 \leftrightarrow \psi_2$ corresponds to $\widetilde{\Phi}_\pm \to \pm\widetilde{\Phi}_\pm$. The fields $\widetilde{\Phi}_\pm$ are sine-Gordon fields for all values of the coupling constant

$$g^2 < \frac{\pi}{\sqrt{2}}, \tag{3.54}$$

for which $D < 2$ and the model is well defined in the sense of a perturbation theory around the scale invariant fixed point.

4. Operator solution and Hilbert space

Now, we are in order to reconstruct from the functional integral bosonization the operator solution for the quantum equations of motion. To begin with, let us consider the vector fields (3.22) which can be written as

$$\mathcal{B}_j^\mu = \mathcal{B}^\mu + \sum_{i_D} \lambda_{jj}^{i_D} \mathcal{B}_{i_D}^\mu. \tag{4.1}$$

In the same way as in the standard Thirring model [9], the auxiliary vector fields are related to the vector currents. Performing the field scaling (3.35)–(3.38) and the canonical transformations (3.40)–(3.44), we obtain

$$\mathcal{B}^\mu = -(N-1)\mathcal{J}^\mu + \ell^\mu, \tag{4.2}$$

$$\mathcal{B}_{i_D}^\mu = \mathcal{J}_{i_D}^\mu + \ell_{i_D}^\mu, \tag{4.3}$$

where the currents are given by

$$\mathcal{J}^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\widetilde{\Phi}, \tag{4.4}$$

$$\mathcal{J}_{i_D}^\mu = -\frac{\gamma}{2\pi}\epsilon^{\mu\nu}\partial_\nu\widetilde{\Phi}_{i_D}. \tag{4.5}$$

Due to the opposite metric for the free massless fields, the currents

$$\ell^\mu = \frac{1}{g}\sqrt{\frac{N-1}{N}}\partial^\mu(\zeta' - \xi) = \partial^\mu \ell, \tag{4.6}$$

$$\ell_{i_D}^\mu = \frac{1}{g}\partial^\mu(\zeta_{i_D}' + \xi_{i_D}) = \partial^\mu \ell_{i_D}, \tag{4.7}$$

are longitudinal currents of zero norm

$$\langle 0|\ell^\mu(x)\ell^\nu(y)|0\rangle = 0 \qquad \forall\, (x,y), \tag{4.8}$$

$$\langle 0|\ell_{i_D}^\mu(x)\ell_{i_D}^\nu(y)|0\rangle = 0 \qquad \forall\, (x,y). \tag{4.9}$$

Following the same procedure, the Fermi field operators (3.24) can be written in terms of Mandelstam operators [25] as follows:

$$\psi_j(x) = \left(\Psi(x)\prod_{i_D=1}^{N-1}\Psi_j^{i_D}(x)\right)\left(\omega(x)\prod_{i_D=1}^{N-1}\omega_j^{i_D}(x)\right), \tag{4.10}$$

where

$$\Psi(x) = \left(\frac{\mu_o^{2d}}{2\pi}\right)^{\frac{1}{2}} :\mathrm{e}^{\mathrm{i}\left(\frac{\beta}{2}\gamma^5\widetilde{\Phi}(x) + \frac{2\pi}{N\beta}\int_{x^1}^\infty \partial_0\widetilde{\Phi}(x^0,z^1)\mathrm{d}z^1\right)}:, \tag{4.11}$$

$$\Psi_j^{i_D}(x) = :\mathrm{e}^{\mathrm{i}\lambda_{jj}^{i_D}\left(\frac{\gamma}{2}\gamma^5\widetilde{\Phi}_{i_D}(x) + \frac{2\pi}{\gamma}\int_{x^1}^\infty \partial_0\widetilde{\Phi}_{i_D}(x^0,z^1)\mathrm{d}z^1\right)}:, \tag{4.12}$$

$$\omega(x) = :\mathrm{e}^{\mathrm{i}g\sqrt{\frac{N-1}{N}}(\zeta'(x) - \xi(x))}:, \tag{4.13}$$

$$\omega_j^{i_D}(x) = :\mathrm{e}^{\mathrm{i}g\lambda_{jj}^{i_D}(\zeta_{i_D}'(x) + \xi_{i_D}(x))}:. \tag{4.14}$$

The field operators (4.11) and (4.12) generalize those expressions obtained by Halpern in [17] in the case of the baryon-number current–current interaction with $N = 2$.

Within the functional integral formulation, the Hilbert space of states is constructed from the generating functional

$$\mathcal{Z}\left[\theta_1,\ldots,\theta_N,\bar\theta_1,\ldots,\bar\theta_N\right] = \int \prod_{j=1}^{N} \mathcal{D}\bar\psi_j \mathcal{D}\psi_j\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^2 z\{\mathcal{L}+\sum_{j=1}^{N}(\bar\theta_j\psi_j+\bar\psi_j\theta_j)\}}, \tag{4.15}$$

whose bosonized form is given by

$$\mathcal{Z}\left[\theta_1,\ldots,\theta_N,\bar\theta_1,\ldots,\bar\theta_N\right] = \left\langle \mathrm{e}^{\mathrm{i}\int \mathrm{d}^2 z\{\sum_{j=1}^{N}\bar\theta_j(\Psi(z)\prod_{i_D}\Psi_j^{i_D}(z)\omega(z)\prod_{i_D}\omega_j^{i_D}(z))+\mathrm{h.c.}\}}\right\rangle, \tag{4.16}$$

where the average is taken with respect to the functional measure

$$d\mu = \mathcal{D}\widetilde{\xi}\, \mathrm{e}^{\mathrm{i}S^{(0)}(\widetilde{\xi})}\mathcal{D}\zeta'\, \mathrm{e}^{-\mathrm{i}S^{(0)}(\zeta')} \prod_{i_D=1}^{N-1}\mathcal{D}\widetilde{\xi}_{i_D}\, \mathrm{e}^{-\mathrm{i}S^{(0)}(\widetilde{\xi}_{i_D})} \prod_{i_D=1}^{N-1}\mathcal{D}\zeta_{i_D}\, \mathrm{e}^{\mathrm{i}S^{(0)}(\zeta_{i_D})}\mathcal{D}\widetilde{\Phi} \prod_{i_D=1}^{N-1}\mathcal{D}\widetilde{\Phi}_{i_D}\, \mathrm{e}^{\mathrm{i}S(\widetilde{\Phi},\widetilde{\Phi}_{i_D})}. \tag{4.17}$$

From the generating functional (4.16) we obtain the general $2n$-point functions, as for instance

$$\begin{aligned}\langle\bar\psi_j(x_1)\cdots\bar\psi_j(x_n)\psi_j(y_1)\cdots\psi_j(y_n)\rangle &= \langle 0|\bar\Psi(x_1)\prod_{i_D}\bar\Psi_j^{i_D}(x_1)\cdots\bar\Psi(x_n)\prod_{i_D}\bar\Psi_j^{i_D}(x_n)\Psi(y_1)\\ &\quad\times\prod_{i_D}\Psi_j^{i_D}(y_1)\cdots\Psi(y_n)\prod_{i_D}\Psi_j^{i_D}(y_n)|0\rangle\langle 0|\omega^*(x_1)\prod_{i_D}\omega_j^{i_D*}(x_1)\cdots\omega^*(x_n)\\ &\quad\times\prod_{i_D}\omega_j^{i_D*}(x_n)\omega(y_1)\prod_{i_D}\omega_j^{i_D}(y_1)\cdots\omega(y_n)\prod_{i_D}\omega_j^{i_D}(y_n)|0\rangle_o,\end{aligned} \tag{4.18}$$

where the notation $\langle 0|\bullet|0\rangle$ means average with respect to the coupled sine-Gordon theories and $\langle 0|\bullet|0\rangle_o$ means average with respect to the free massless theories. Due to the opposite metric quantization for the free massless fields, the operators ω and $\omega_j^{i_D}$ generate constant Wightman functions

$$\langle 0|\omega^*(x_1)\cdots\omega^*(x_n)\omega(y_1)\cdots\omega(y_n)|0\rangle_o = 1, \tag{4.19}$$

$$\langle 0|\omega_j^{i_D*}(x_1)\cdots\omega_j^{i_D*}(x_n)\omega_j^{i_D}(y_1)\cdots\omega_j^{i_D}(y_n)|0\rangle_o = 1. \tag{4.20}$$

The cluster decomposition is not violated, since the operators ω and $\omega_j^{i_D}$ carry neither fermionic charge nor chirality selection rules. The operators ω and $\omega_j^{i_D}$ commute with all operators defining the field algebra and thus reduce to the identity in the Hilbert space of states. Following [13], the states in the positive semi-definite Hilbert space $\mathcal{H}$ can be accommodated as equivalent classes modulo $\ell_\mu|0\rangle$ and $\ell_\mu^{i_D}|0\rangle$, in such a way that the positive-definite metric Hilbert space $\mathcal{H}'$ of the model is a proper subspace of $\mathcal{H}$ and is given by the coset

$$\mathcal{H}' = \frac{\mathcal{H}}{h_o}, \tag{4.21}$$

where h_o is the zero norm Hilbert subspace generated by the longitudinal currents ℓ_μ and $\ell_\mu^{i_D}$.

The Fermi field (4.10) satisfies the quantum version of the equation of motion (2.5)

$$\mathrm{i}\gamma^\mu\partial_\mu\psi_j(x) = -g^2(N-1)\,\vdots\mathcal{J}_\mu(x)\gamma^\mu\psi_j(x)\vdots + g^2\sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\vdots\mathcal{J}_\mu^{i_D}(x)\gamma^\mu\psi_j(x)\vdots - M\psi_j(x), \tag{4.22}$$

where $\vdots(\bullet)\vdots$ denotes a suitable defined normal product given by the symmetric limit

$$\vdots A(x)B(x)\vdots \doteq \lim_{\varepsilon\to 0} \tfrac{1}{2}\left\{A(x+\varepsilon)B(x) + A(x-\varepsilon)B(x)\right\}, \tag{4.23}$$

and the currents are given by (4.4) and (4.5). In equation (4.22), M is a constant which is infinite, where the scale dimension of the mass operator (3.49) is greater than zero; finite, where it is equal to zero; or zero, where that scale is less than one [1, 26]. For $g = 0$ we obtain the canonical value $D = 1$.

For $g = 0$ and using (3.50), the operator solutions (4.10) correspond to the Mandelstam soliton operators describing N free massive Fermi fields

$$\Psi_j(x) \equiv \Psi_j^{(0)} = \left(\frac{\mu_o}{2\pi}\right)^{\frac{1}{2}} \boldsymbol{:}\, \mathrm{e}^{\mathrm{i}\sqrt{2\pi}(\gamma^5\widetilde{\Phi}_j(x) + \int_{x^1}^{\infty}\partial_0\widetilde{\Phi}_j(x^0,z^1)\mathrm{d}z^1)}\boldsymbol{:}. \tag{4.24}$$

The exact general Wightman functions of the model with massless fermions are presented in appendix A.

5. Local gauge symmetry

In this section we shall consider the model with local $U(1)$ gauge symmetry. This will enable us to have a clear understanding of the mathematical criteria within the functional integral bosonization framework, in order to deal with 'decoupled' free massless fields when constructing the physical Hilbert subspace of states in the case of a more complex situation.

To begin with, let us consider QED$_2$ with N flavored Fermi fields with quartic interaction among different fermion species, which is defined by the Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \sum_{j=1}^{N} \bar{\psi}_j(x)(\mathrm{i}\gamma^\mu\partial_\mu + e\gamma^\mu\mathcal{A}_\mu - m_o)\psi_j(x)$$
$$+ g^2\sum_{j<k}^{N}(\bar{\psi}_j(x)\gamma^\mu\psi_j(x))(\bar{\psi}_k(x)\gamma_\mu\psi_k(x)), \tag{5.1}$$

where the field-strength tensor is given by

$$\mathcal{F}_{\mu\nu} = \partial_\nu\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\nu. \tag{5.2}$$

Following the same procedure as in section 3, after decoupling the quartic interactions, the Lagrangian (5.1) can be written as

$$\mathcal{L}(x) = -\frac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \sum_{j=1}^{N} \bar{\psi}_j(x)(\mathrm{i}\gamma^\mu\partial_\mu + \gamma^\mu(g^2\mathcal{B}_\mu + e\mathcal{A}_\mu) - m_o)\psi_j(x)$$
$$+ \frac{g^2}{2}\sum_{j=1}^{N}(\mathcal{B}_j^\mu)^2 - \frac{1}{2}\frac{g^2}{(N-1)}\left(\sum_{j=1}^{N}\mathcal{B}_j^\mu\right)^2. \tag{5.3}$$

In order to decouple the Fermi fields and the vector fields $\mathcal{B}_\mu$ and $\mathcal{A}_\mu$ in the effective Lagrangian, we shall use the parametrization (3.10) for the fields $\mathcal{B}_\pm^j$ and the fields $\mathcal{A}_\pm$ are parametrized as

$$\mathcal{A}_+ = \frac{1}{e}U_g^{-1}\mathrm{i}\partial_+U_g, \qquad \mathcal{A}_- = \frac{1}{e}V_g\mathrm{i}\partial_-V_g^{-1}. \tag{5.4}$$

The label 'g' characterizes the fields associated with the gauge degrees of freedom. The decoupling is achieved by performing the simultaneous fermion rotations

$$\psi_{(1)j} = V_jV_g\psi_{(1)j}^{(0)}, \qquad \psi_{(2)j} = U_j^{-1}U_g^{-1}\psi_{(2)j}^{(0)}. \tag{5.5}$$

The effective bosonized Lagrangian is given by (for details see appendix B)

$$\mathcal{L}_{\text{eff}} = \delta\mathcal{L}_o + \widehat{\mathcal{L}}, \tag{5.6}$$

where

$$\delta\mathcal{L}_o = -\frac{1}{2}(\partial_\mu\zeta')^2 + \frac{1}{2}(\partial_\mu\widetilde{\xi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\zeta_{i_D}\right)^2 - \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\xi}_{i_D}\right)^2 \tag{5.7}$$

and

$$\widehat{\mathcal{L}} = -\frac{1}{2}(\partial_\mu\widetilde{\eta})^2 + \frac{1}{2}(\partial_\mu\widetilde{\Phi})^2 + \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu\widetilde{\Phi}_{i_D}\right)^2 + \frac{1}{2}(\partial_\mu\widetilde{\Sigma})^2 - \frac{1}{2}m^2\widetilde{\Sigma}^2$$
$$- m'_o\sum_{j=1}^{N}\cos\left(\beta\widetilde{\Sigma} + \beta(\widetilde{\Phi}+\widetilde{\eta}) + \gamma\sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\widetilde{\Phi}_{i_D}\right), \tag{5.8}$$

with

$$\beta^2 = \frac{4\pi}{N}\frac{1}{1-\frac{g^2}{\pi}(N-1)}, \tag{5.9}$$

$$\gamma^2 = \frac{4\pi}{1+\frac{g^2}{\pi}}, \tag{5.10}$$

and the mass of the 'physical' field $\widetilde{\Sigma}$ is given by

$$m^2 = \frac{Ne^2}{\pi}\frac{1}{1-\frac{g^2}{\pi}(N-1)}. \tag{5.11}$$

5.1. Operator solution, gauge invariant field algebra and Hilbert space hierarchy

Following the same procedure given in section 4, the auxiliary vector field $\mathcal{B}_j^\mu$ (3.22) and the gauge field $\mathcal{A}_\mu$ (4.16) can be written as

$$\mathcal{A}_\mu = \frac{1}{m}\epsilon_{\mu\nu}\partial^\nu(\widetilde{\Sigma}+\widetilde{\eta}), \tag{5.12}$$

$$\mathcal{B}_j^\mu = \mathcal{B}^\mu + \sum_{i_D}\lambda_{jj}^{i_D}\mathcal{B}_{i_D}^\mu, \tag{5.13}$$

where

$$\mathcal{B}^\mu = -(N-1)\mathcal{J}^\mu + L^\mu + \ell^\mu, \tag{5.14}$$

$$\mathcal{B}_{i_D}^\mu = \mathcal{J}_{i_D}^\mu + \ell_{i_D}^\mu, \tag{5.15}$$

$$\mathcal{J}^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\widetilde{\Sigma}, \tag{5.16}$$

$$\mathcal{J}_{i_D}^\mu = -\frac{\gamma}{2\pi}\epsilon^{\mu\nu}\partial_\nu\widetilde{\Phi}_{i_D}, \tag{5.17}$$

and the longitudinal currents of zero norm are given by

$$L_\mu = (N-1)\frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu(\widetilde{\eta}+\widetilde{\Phi}), \tag{5.18}$$

$$\ell_\mu = \frac{1}{g}\sqrt{\frac{N-1}{N}}\partial_\mu(\zeta' - \xi), \tag{5.19}$$

$$\ell^\mu_{i_D} = \frac{1}{g}\partial^\mu\left(\xi_{i_D} + \zeta'_{i_D}\right). \tag{5.20}$$

The Fermi field operator is given by (see equation (4.10))

$$\psi_j = :\mathrm{e}^{\mathrm{i}\gamma^5\frac{\beta}{2}(\widetilde{\Sigma}+\widetilde{\eta})}:\left(\Psi(x)\prod_{i_D=1}^{N-1}\Psi_j^{i_D}(x)\right)\left(\omega(x)\prod_{i_D=1}^{N-1}\omega_j^{i_D}(x)\right). \tag{5.21}$$

The gauge non-invariant indefinite-metric Hilbert space $\mathcal{H}$ contains zero norm states generated by the currents ℓ^μ, $\ell^\mu_{i_D}$ and L^μ. The positive-norm Hilbert subspace $\mathcal{H}''$ is the quotient space

$$\mathcal{H}'' = \frac{\mathcal{H}}{\mathcal{H}_o}, \tag{5.22}$$

where $\mathcal{H}_o$ is the zero norm subspace generated by the currents ℓ_μ, $\ell^{i_D}_{\mu j}$ and L_μ. In this subspace the operators ω and $\omega_j^{i_D}$ reduce to the identity.

The gauge invariant field subalgebra is generated by the set of local operators $\left\{\mathcal{J}^\mu, \mathcal{J}^\mu_{i_D}, \mathcal{F}_{\mu\nu}\right\}$ and the bilocals formally defined by

$$D_j(x,y) \sim \psi_j^\dagger(x)\,\mathrm{e}^{-\mathrm{i}e\int_x^y \mathcal{A}_\mu \mathrm{d}z^\mu}\psi_j(y). \tag{5.23}$$

Using the independence of the choice of integration-path (up to a c-number phase) of the exponential [28, 20, 19], the bilocal operators are given by (up to a normalization factor)

$$D_j(x,y) = :\mathrm{e}^{\mathrm{i}\{\frac{\beta}{2}(\gamma_y^5\widetilde{\Sigma}(y)-\gamma_x^5\widetilde{\Sigma}(x))-\frac{2\pi}{N\beta}\int_x^y \epsilon_{\mu\nu}\partial^\nu\widetilde{\Sigma}(z)\mathrm{d}z^\mu\}}: \times\left(\prod_{i_D=1}^{N-1}:\Psi_j^{i_D *}(x)\Psi_j^{i_D}(y):\right)\sigma^*(x)\sigma(y), \tag{5.24}$$

where

$$\sigma(x) = :\mathrm{e}^{\mathrm{i}\{\gamma^5\frac{\beta}{2}(\widetilde{\Phi}(x)+\widetilde{\eta}(x))+\frac{2\pi}{N\beta}\int_x^\infty \epsilon_{\mu\nu}\partial^\nu(\widetilde{\Phi}(z)+\widetilde{\eta}(z))\mathrm{d}z^\mu\}}:. \tag{5.25}$$

The operator (5.25) is a generalization of the 'constant unitary operator' obtained by Lowenstein and Swieca [18] for QED_2. As in the standard QED_2 [19, 18], the cluster decomposition is violated, since the operator $\sigma_{(\alpha)}$ ($\alpha = 1, 2$ refers to spinor index) generates an infinite number of vacua carrying the $U(1)$ charge and chirality of the Thirring Fermi field Ψ (4.11),

$$\sigma_{(1)}^{n_1}\sigma_{(2)}^{n_2}|0\rangle = |n_1, n_2\rangle. \tag{5.26}$$

The cluster decomposition is restored by introducing the coherent superposition [18]

$$|\theta_1,\theta_2\rangle = \frac{1}{2\pi}\sum_{n_1,n_2=0}^{\infty}\mathrm{e}^{-\mathrm{i}n_1\theta_1}\mathrm{e}^{-\mathrm{i}n_2\theta_2}|n_1,n_2\rangle, \tag{5.27}$$

such that

$$\sigma_{(\alpha)}|\theta_1,\theta_2\rangle = \mathrm{e}^{\mathrm{i}\theta_\alpha}|\theta_1,\theta_2\rangle. \tag{5.28}$$

In each one of the irreducible sectors $\sigma_{(1)}$ and $\sigma_{(2)}$ are c-numbers

$$\sigma_{(1)}^*\sigma_{(2)} = \mathrm{e}^{-\mathrm{i}(\theta_2-\theta_1)}, \tag{5.29}$$

and the field algebra is isomorphic with the algebra of the sine-Gordon fields $\widetilde{\Sigma}$ and $\widetilde{\Phi}_{i_D}$. Within the functional integral approach, the generating functional of the gauge invariant Wightman functions is given in terms of the bosonized theory defined by the Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\sum_{i_D=1}^{N-1}\left(\partial_\mu \widetilde{\Phi}_{i_D}\right)^2 + \frac{1}{2}(\partial_\mu \widetilde{\Sigma})^2 - \frac{1}{2}m^2\widetilde{\Sigma}^2 - m'_o\sum_{j=1}^{N}\cos\left(\beta\widetilde{\Sigma} + \gamma\sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\widetilde{\Phi}_{i_D} + \theta\right), \tag{5.30}$$

where $\theta = \theta_2 - \theta_1$ characterizes the explicity chiral symmetry breakdown. For $g^2 = 0\,(\beta^2 = \frac{4\pi}{N})$ we recover from (5.21) the operator solution obtained in [19, 1] for QED_2 with N flavored Fermi fields, and from (5.30) the corresponding effective bosonized Lagrangian.

6. Statistical mechanical description

It is well known that in the standard Thirring model the scale dimension of the mass operator must be $D < 2$. The same must occur in the theory described by the Lagrangian (2.1) when the short distance fixed point is defined within the model with massless fermions. In what follows we shall consider the statistical mechanical system associated with the effective bosonized theory and show that, analogously to the standard Thirring model, in the present case the equation of state exhibits a Kosterlitz–Thouless phase transition at the critical dimension $D_c = 2$.

In the two-dimensional Euclidean space the partition function is given by (we shall suppress the 'tilde' notation)

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_o}\int d\boldsymbol{\mu}_o \exp\left\{-m'_o\int d^2z\sum_{j=1}^{N}\cos\left(\beta\Phi(z) + \gamma\sum_{i_D}\lambda_{jj}^{i_D}\Phi_{i_D}(z)\right)\right\}, \tag{6.1}$$

where $d\boldsymbol{\mu}_o$ is the free-field (Gaussian) probability measure

$$d\boldsymbol{\mu}_o = \mathcal{D}\Phi\, e^{-S^{(0)}(\Phi)}\prod_{i_D=1}^{N-1}\mathcal{D}\Phi_{i_D}\, e^{-S^{(0)}_{i_D}(\Phi_{i_D})}, \tag{6.2}$$

the $S^{(0)}$'s are the corresponding free-field Euclidian actions, $\mathcal{D}\Phi$ is the formal Lebesgue measure and

$$\mathcal{Z}_o = \int d\boldsymbol{\mu}_o. \tag{6.3}$$

By expanding the exponential of the interaction action in the Gell'Mann–Low formula (6.1) in a power series of the bare mass m'_o, the interaction term can be treated as a perturbation in the corresponding free-field theories defined by the actions $S^{(0)}$.

6.1. Partition function

Following the standard procedure [13, 21], expanding the exponentials in (6.1) in powers of m'_o and performing the functional integration over the fields Φ and Φ_{i_D} (for more details see appendix C), we get

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_o} \sum_{n=0}^{\infty} \frac{(-m_o')^n}{2^n} \sum_{m_1,m_2,\ldots,m_N} \frac{\delta_{n,m_1+\cdots+m_N}}{m_1! m_2! \cdots m_N!} \left(\prod_{j=1}^{N} \int \prod_{k_j=1}^{m_j} \mathrm{d}^2 z_{k_j} \right)$$
$$\times \sum_{\{\alpha_{k_j}\}_{m_j}} \exp\left\{ -\frac{\beta^2}{2} \sum_{j=1}^{N} \sum_{j'=1}^{N} \sum_{k_j=1}^{m_j} \sum_{\bar{k}_{j'}=1}^{m_{j'}} \alpha_{k_j} \alpha_{\bar{k}_{j'}} D_o(z_{k_j} - z_{\bar{k}_{j'}}) \right\}$$
$$\times \exp\left\{ -\frac{\gamma^2}{2} \sum_{i_D=1}^{N-1} \sum_{j=1}^{N} \sum_{j'=1}^{N} \sum_{k_j=1}^{m_j} \sum_{\bar{k}_{j'}=1}^{m_{j'}} \alpha_{k_j} \alpha_{\bar{k}_{j'}} \lambda_{jj}^{i_D} \lambda_{j'j'}^{i_D} D_o(z_{k_j} - z_{\bar{k}_{j'}}) \right\}, \tag{6.4}$$

where $\alpha_{k_j} = \pm 1$, the summation $\sum_{\{\alpha_{k_j}\}_{m_j}}$ runs over all possibilities in the set $\{\alpha_1, \ldots, \alpha_{m_j}\}$, the summation $\sum_{m_1,m_2,\ldots,m_N}$ runs over all positive-integer values of m_j for which

$$\sum_{j=1}^{N} m_j = n, \tag{6.5}$$

and

$$D_o(z) = \lim_{\mu^2 \to 0} \Delta(z; \mu) = -\frac{1}{4\pi} \ln\{-\mu^2(|z|^2 + \varepsilon^2)\} \tag{6.6}$$

is the infrared and ultraviolet regularized massless Green's function of the two-dimensional Laplacian operator. We carry out the calculations in the presence of μ^2 and set $\mu^2 \to 0$ at the end. The contributions of the infrared cut-off μ^2 in equation (6.4) factorize and are given by

$$f(\mu^2) = (\mu^2)^{\frac{\beta^2}{8\pi} (\sum_{j=1}^{N} \sum_{k_j=1}^{m_j} \alpha_{k_j})^2} (\mu^2)^{\frac{\gamma^2}{8\pi} \sum_{i_D=1}^{N-1} (\sum_{j=1}^{N} \sum_{k_j=1}^{m_j} \alpha_{k_j} \lambda_{jj}^{i_D})^2}. \tag{6.7}$$

In the limit $\mu^2 \to 0$ the non-zero contributions in the partition function (6.4) are those with m_j even that satisfy the selection rule

$$\sum_{k_j=1}^{m_j} \alpha_{k_j} = 0. \tag{6.8}$$

The selection rule (6.8) together with condition (6.5) also imply that the non-zero contributions in the partition function (6.4) are those with n even.

For $j = j'$ and $k_j = \bar{k}_j$, the ε-dependent terms in equation (6.4), corresponding to the self-energy of the charges in the gas factorize

$$\prod_{j=1}^{N} (|\varepsilon|^2)^{\frac{2m_j}{8\pi}(\beta^2 + \gamma^2 \frac{(N-1)}{N})} = (|\varepsilon|^2)^{\frac{2n}{8\pi}(\beta^2 + \gamma^2 \frac{(N-1)}{N})}, \tag{6.9}$$

and can be removed by the 'fugacity' renormalization

$$\boldsymbol{z} = \frac{m_o'}{2} (|\varepsilon|^2)^{\frac{1}{8\pi}(\beta^2 + \gamma^2 \frac{(N-1)}{N})}. \tag{6.10}$$

The partition function corresponds to a 'multi-gas' expansion and is given by

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_o} \sum_{n=0}^{\infty} \boldsymbol{z}^{2n} \sum_{2m_1,2m_2,\ldots,2m_N} \frac{\delta_{2n,2m_1+\cdots+2m_N}}{2m_1! 2m_2! \cdots 2m_N!} \left(\prod_{j=1}^{N} \int \prod_{k_j=1}^{2m_j} \mathrm{d}^2 z_{k_j} \right)$$
$$\times \sum_{\{\alpha_{k_j}\}_{2m_j}} \exp\left\{ \frac{1}{8\pi} \left(\beta^2 + \frac{\gamma^2(N-1)}{N} \right) \sum_{j=1}^{N} \sum_{k_j \neq \bar{k}_j}^{2m_j} \alpha_{k_j} \alpha_{\bar{k}_j} \ln\left(|z_{k_j} - z_{\bar{k}_j}|^2 + |\varepsilon|^2\right) \right\}$$
$$\times \exp\left\{ \frac{1}{8\pi} \left(\beta^2 - \frac{\gamma^2}{N} \right) \sum_{j \neq j'}^{N} \sum_{k_j=1}^{2m_j} \sum_{\bar{k}_{j'}=1}^{2m_{j'}} \alpha_{k_j} \alpha_{\bar{k}_{j'}} \ln\left(|z_{k_j} - z_{\bar{k}_{j'}}|^2 + |\varepsilon|^2\right) \right\}. \tag{6.11}$$

The first exponential in (6.11) corresponds to the interaction energy between the charges in the gas associated with the same Fermi field species. The second exponential in (6.11) is due to the interaction energy between the charges in the gas associated with different Fermi field species. For $g = 0\,(\beta^2 = 4\pi/N, \gamma^2 = 4\pi)$ this term disappears and the partition function factorizes into a product of N partition functions describing the statistical mechanical system of N independent free massive Fermi field theories

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_o}\sum_{n=0}^{\infty} z^{2n} \sum_{2m_1,2m_2,\ldots,2m_N} \frac{\delta_{2n,2m_1+\cdots+2m_N}}{2m_1!2m_2!\cdots 2m_N!}\prod_{j=1}^{N}\mathcal{Z}^{(2m_j)}, \tag{6.12}$$

where

$$\mathcal{Z}^{(2m_j)} = \int \prod_{k_j=1}^{2m_j} d^2 z_{k_j} \sum_{\{\alpha_{k_j}\}_{2m_j}} \exp\left\{\frac{1}{2}\sum_{k_j\neq\bar{k}_j}^{2m_j} \alpha_{k_j}\alpha_{\bar{k}_j} \ln\left(|z_{k_j} - z_{\bar{k}_j}|^2 + |\varepsilon|^2\right)\right\}. \tag{6.13}$$

6.2. Equation of state

Following the standard procedure [13, 21], in order to obtain the equation of state of the statistical mechanical system described by the partition function (6.11), we shall consider the system confined in a finite volume $\mathcal{V} = \pi R^2$. The thermodynamical limit is performed at the end of all calculations. In order to extract the volume dependence in the partition function, we shall make the change of variables $z_{k_j} \to \hat{z}_{k_j} = z_{k_j}/R$. Using the selection rule (6.8) and the constraint (6.5), the partition function (6.11) can be written as

$$\begin{aligned}\mathcal{Z} = {} & \frac{1}{\mathcal{Z}_o}\sum_{n=0}^{\infty} z^{2n}\left(\frac{\mathcal{V}}{\pi}\right)^{2n\left[1-\frac{1}{8\pi}\left(\beta^2+\gamma^2\frac{(N-1)}{N}\right)\right]} \\ & \times \sum_{2m_1,2m_2,\ldots,2m_N} \frac{\delta_{2n,2m_1+\cdots+2m_N}}{2m_1!2m_2!\cdots 2m_N!}\left(\prod_{j=1}^{N}\int_{|\hat{z}_k|<1}\prod_{k_j=1}^{2m_j} \mathrm{d}^2\hat{z}_{k_j}\right) \\ & \times \sum_{\{\alpha_{k_j}\}_{2m_j}} \exp\left\{\frac{1}{8\pi}\left(\beta^2 + \frac{\gamma^2(N-1)}{N}\right)\sum_{j=1}^{N}\sum_{k_j\neq\bar{k}_j}^{2m_j}\alpha_{k_j}\alpha_{\bar{k}_j}\ln\left(|\hat{z}_{k_j} - \hat{z}_{\bar{k}_j}|^2 + |\hat{\varepsilon}|^2\right)\right\} \\ & \times \exp\left\{\frac{1}{8\pi}\left(\beta^2 - \frac{\gamma^2}{N}\right)\sum_{j\neq j'}^{N}\sum_{k_j=1}^{2m_j}\sum_{\bar{k}_{j'}=1}^{2m_{j'}}\alpha_{k_j}\alpha_{\bar{k}_{j'}}\ln\left(|\hat{z}_{k_j} - \hat{z}_{\bar{k}_{j'}}|^2 + |\hat{\varepsilon}|^2\right)\right\}.\end{aligned} \tag{6.14}$$

Introducing the potential

$$\Omega = -KT\ln\mathcal{Z}, \tag{6.15}$$

the pressure is given by

$$\mathcal{P} = -\left(\frac{\partial\Omega}{\partial\mathcal{Z}}\right) = KT\frac{1}{\mathcal{Z}}\left(\frac{\partial\mathcal{Z}}{\partial\mathcal{V}}\right). \tag{6.16}$$

The variation of (6.14) with respect to the volume leads to the following equation of state:

$$\mathcal{P}\mathcal{V} = \left(1 - \frac{D}{2}\right)\langle\mathcal{N}\rangle KT, \tag{6.17}$$

where D is the scale dimension of the mass operator (3.49) and $\langle\mathcal{N}\rangle$ is the expected number of particles,

$$\langle\mathcal{N}\rangle = \frac{1}{\mathcal{Z}}\sum_{n=0}^{\infty} z^{2n}(2n)\left(\frac{\mathcal{V}}{\pi}\right)^{2n[1-\frac{1}{8\pi}(\beta^2+\gamma^2\frac{(N-1)}{N})]}$$
$$\times \sum_{2m_1,2m_2,\ldots,2m_N}\frac{\delta_{2n,2m_1+\cdots+2m_N}}{2m_1!2m_2!\cdots 2m_N!}\left(\prod_{j=1}^{N}\int_{|z_k|<1}\prod_{k_j=1}^{2m_j}\mathrm{d}^2\hat{z}_{k_j}\right)$$
$$\times \sum_{\{\alpha_{k_j}\}_{2m_j}}\exp\left\{\frac{1}{8\pi}\left(\beta^2+\frac{\gamma^2(N-1)}{N}\right)\sum_{j=1}^{N}\sum_{k_j\neq\bar{k}_j}^{2m_j}\alpha_{k_j}\alpha_{\bar{k}_j}\ln\left(|\hat{z}_{k_j}-\hat{z}_{\bar{k}_j}|^2+|\hat{\varepsilon}|^2\right)\right\}$$
$$\times \exp\left\{\frac{1}{8\pi}\left(\beta^2-\frac{\gamma^2}{N}\right)\sum_{j\neq j'}^{N}\sum_{k_j=1}^{2m_j}\sum_{\bar{k}_{j'}=1}^{2m_{j'}}\alpha_{k_j}\alpha_{\bar{k}_{j'}}\ln\left(|\hat{z}_{k_j}-\hat{z}_{\bar{k}_{j'}}|^2+|\hat{\varepsilon}|^2\right)\right\}. \tag{6.18}$$

The equation of state (6.17) exhibits a Kosterlitz–Thoulesss phase transition [22, 27] at the critical value

$$D_c = 2, \tag{6.19}$$

that is,

$$\beta^2 + \frac{\gamma^2(N-1)}{N} = 8\pi. \tag{6.20}$$

The equation of state (6.17) describes the behavior of the statistical mechanical system defined by the partition function (6.14), which is associated with the bosonic theory defined by the Lagrangian (3.48). However, one may ensure that the Lagrangian (3.48) corresponds to the bosonized version of the fermionic model only for a mass operator with scale dimension $D < 2$, such that for short distances the mass perturbation becomes increasingly negligible. In the critical region of the equation of state, the inequality (3.49) is violated so that for short distances the model starts to be driven away from the fixed point and thus cannot be considered as a perturbation in the scale invariant massless fermionic model. The statistical mechanical system associated with the effective bosonized theory describing the original fermionic model is restricted to the region $D < 2$ and the critical point for the K–T phase transition lies outside of the domain where the fermion–boson mapping can be established.

For $N = 2$ the equation of state (6.17) reduces to that obtained in [13]. For $g = 0$ the equation of state (6.17) is given by

$$\mathcal{P}\mathcal{V} = \tfrac{1}{2}\langle\mathcal{N}\rangle KT, \tag{6.21}$$

and corresponds to the equation of state of the standard Coulomb gas [27].

7. Concluding remarks

Using the Abelian reduction of the WZW theory, we have considered the functional integral bosonization of the two-dimensional fermion model with Thirring interaction among different species. In the same way as in the operator approach developed in [17], in the present functional integral approach we only need to know the free-field fermion–boson correspondences in order to perform the bosonization of the interacting theory and the operator solution for the quantum equations of motion. The use of auxiliary vector fields introduces redundant bosonic degrees of freedom which are not intrinsic fields describing the physical content of the model. The resulting bosonized model is defined in a positive-metric Hilbert space and corresponds to N coupled sine-Gordon fields $\{\widetilde{\Phi}, \widetilde{\Phi}_{i_D}\}$. The operator solution for the quantum equations of motion is constructed from the functional integral approach and is given in terms of generalized Mandelstam soliton operators. The fermion–boson correspondences obtained by Halpern [17] for the $U(1)$ current–current interaction are generalized to the Thirring interaction between different fermion species. The extension of the model to local gauge symmetry was discussed and the $U(1)$ charge screening mechanism of the Thirring field is presented. In this case the gauge non-invariant indefinite metric Hilbert space exhibits two free massless scalar excitations, one coming from the use of auxiliary vector fields and the other from the gauge structure of the model, each one of them playing a distinct role in the definition of the physical content of the model. This generalizes the presentation of [1, 13, 19].

The statistical mechanical description of the effective bosonized theory was performed by obtaining the corresponding partition function and the equation of state. As in the standard Thirring model, the equation of state exhibits a K–T phase transition at the critical dimension $D_c = 2$.

We conclude that in order to exert control on the effect of the redundant Bose fields, introduced by the use of auxiliary vector fields, and to obtain the fermion–boson mapping in the Hilbert space of states, the functional integral bosonization must be performed on the generating functional—and not on the partition function. Although the massless Bose fields give no physical contributions to the Wightman functions, the most appropriate way to treat the problem is to perform the functional integral bosonization of the generating functional of the theory, from which one constructs the Hilbert space of the model, without disregarding the role played by the 'decoupled' massless Bose fields in the intermediary steps. In appendix D, we present our criticism with respect to the conclusions of [3], in which the fermion–boson mapping is established on the partition function level.

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Appendix A. Massless model

For massless fermions ($m_o = 0$) the model is a scale invariant theory with anomalous scale dimension, described by N free massless fields $\widetilde{\Phi}, \widetilde{\Phi}_{i_D}$. In order to compute the general Wightman functions for the massless Fermi fields, let us introduce the light-cone variables

$$u = x^0 + x^1, \qquad v = x^0 - x^1, \tag{A.1}$$

such that the pseudo-scalar and scalar fields are defined in terms of right- and left-mover components by

$$\widetilde{\Phi}(x) = \Phi(v) - \Phi(u), \qquad \Phi(x) = \Phi(v) + \Phi(u). \tag{A.2}$$

In this case, the operator solution (4.10) is written in terms of the free massless fields as follows:

$$\Psi(x) = \left(\frac{\mu^{2d}}{2\pi}\right)^{\frac{1}{2}} :\mathrm{e}^{\mathrm{i}(\frac{2\pi}{N\beta}+\gamma^5\frac{\beta}{2})\Phi(v)}::\mathrm{e}^{\mathrm{i}(\frac{2\pi}{N\beta}-\gamma^5\frac{\beta}{2})\Phi(u)}:, \tag{A.3}$$

$$\widehat{\Psi}_j(x) = :\mathrm{e}^{\mathrm{i}(\frac{2\pi}{\gamma}+\gamma^5\frac{\gamma}{2})\sum_{i_D}\lambda_{jj}^{i_D}\Phi_{i_D}(v)}::\mathrm{e}^{\mathrm{i}(\frac{2\pi}{\gamma}-\gamma^5\frac{\gamma}{2})\sum_{i_D}\lambda_{jj}^{i_D}\Phi_{i_D}(u)}:. \tag{A.4}$$

The general Wightman functions for the same spinor components are given by

$$\langle 0|\psi_j(x_1)\cdots\psi_j(x_n)\psi_j^*(\bar{x}_1)\cdots\psi_j^*(\bar{x}_n)|0\rangle$$
$$= \left(\frac{1}{2\pi}\right)^n \prod_{i<k}^{n}([\mathrm{i}(v_i - v_k)]^{d+\frac{1}{2}\gamma^5}[\mathrm{i}(u_i - u_k)]^{d-\frac{1}{2}\gamma^5})$$
$$\times \prod_{i<k}^{n}([\mathrm{i}(\bar{v}_i - \bar{v}_k)]^{d+\frac{1}{2}\gamma^5}[\mathrm{i}(\bar{u}_i - \bar{u}_k)]^{d-\frac{1}{2}\gamma^5})$$
$$\times \prod_{i,k}^{n}([\mathrm{i}(v_i - \bar{v}_k)]^{-d-\frac{1}{2}\gamma^5}[\mathrm{i}(u_i - \bar{u}_k)]^{-d+\frac{1}{2}\gamma^5}), \qquad \text{(A.5)}$$

where d is the scale dimension of the Fermi field operator

$$d = \left(\frac{\beta^2}{16\pi} + \frac{\pi}{N^2\beta^2}\right) + \frac{(N-1)}{N}\left(\frac{\gamma^2}{16\pi} + \frac{\pi}{\gamma^2}\right). \qquad \text{(A.6)}$$

For $g = 0$, $d = 1/2$, we recover from (A.5) the Wightman functions for N free massless Fermi fields.

Appendix B. Bosonized Lagrangian of the gauge model

Let us define

$$D_\pm(\mathcal{B}^j, \mathcal{A}) = \big(\mathrm{i}\partial_\pm + g^2\mathcal{B}_\pm^j + e\mathcal{A}_\pm\big), \qquad \text{(B.1)}$$

and introducing in the functional integral the $2N$ identities

$$1 = \int \mathcal{D}U_j\mathcal{D}U_g[\det D_+(U_jU_g)]\delta\big(g^2\mathcal{B}_+^j + e\mathcal{A}_+ - U_j^{-1}U_g^{-1}\mathrm{i}\partial_+U_jU_g\big), \qquad \text{(B.2)}$$

$$1 = \int \mathcal{D}V_j\mathcal{D}V_g[\det D_-(V_jV_g)]\delta\big(g^2\mathcal{B}_-^j + e\mathcal{A}_- - V_jV_g\mathrm{i}\partial_-V_j^{-1}V_g^{-1}\big), \qquad \text{(B.3)}$$

the change of variables from $(\mathcal{B}_\pm^j, \mathcal{A}_\pm)$ to (U_jU_g, V_jV_g) is performed by integration over the field components $\mathcal{B}_\pm^j, \mathcal{A}_\pm$. Performing the fermion chiral rotations (5.5), taking into account the corresponding change in the integration measure [9] and using a gauge invariant regularization, we obtain

$$\left(\prod_j \mathcal{D}\bar{\psi}_j\mathcal{D}\psi_j\right)\left(\prod_j \mathcal{D}\mathcal{B}_\pm^j\right)\mathcal{D}\mathcal{A}_\pm$$
$$= \left(\prod_j \mathcal{D}\bar{\psi}_j^{(0)}\mathcal{D}\psi_j^{(0)}\right)\left(\prod_j \mathcal{D}U_j\mathcal{D}V_j\right)\mathcal{D}U_g\mathcal{D}V_g\prod_j \mathcal{J}_j[U_jV_jU_gV_g], \qquad \text{(B.4)}$$

with

$$\mathcal{J}_j[U_jV_jU_gV_g] = \mathrm{e}^{-\mathrm{i}\Gamma[U_jV_jU_gV_g]} = \mathrm{e}^{-\mathrm{i}\Gamma[G_jG_g]}, \qquad \text{(B.5)}$$

where the fields G_j and G_g are defined by

$$G_j = U_jV_j, \qquad G_g = U_gV_g. \qquad \text{(B.6)}$$

Using the decomposition (3.22) for the auxiliary vector field $\mathcal{B}_\mu$ and decomposing the gauge field as

$$\mathcal{A}^\mu = \frac{1}{e}(\epsilon^{\mu\nu}\partial_\nu\widetilde{\phi}_g + \partial^\mu\zeta_g), \qquad \text{(B.7)}$$

the Bose fields (U_j, V_j, U_g, V_g) are given by

$$U_j = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_j+\zeta_j)}, \qquad V_j = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_j-\zeta_j)}, \tag{B.8}$$

$$U_g = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_g+\zeta_g)}, \qquad V_g = \mathrm{e}^{-\mathrm{i}(\widetilde{\phi}_g-\zeta_g)}. \tag{B.9}$$

The Maxwell Lagrangian is given by

$$\mathcal{L}_M = \frac{1}{8\mathrm{e}^2}\left\{\partial_+\left(G_g i\partial_- G_g^{-1}\right)\right\}^2 = \frac{1}{2\mathrm{e}^2}(\Box\widetilde{\phi}_g)^2. \tag{B.10}$$

Using the decompositions (2.7)–(3.32), the bosonized effective Lagrangian can be written as

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 + \frac{1}{2}\sum_{i_D}\left(\partial_\mu\widetilde{\varphi}_{i_D}\right)^2 + \frac{\pi}{2N\mathrm{e}^2}(\Box\widetilde{\phi}'_g)^2 - \frac{1}{2}(\partial_\mu\widetilde{\phi}'_g)^2 + \frac{1}{2}(\partial_\mu\widetilde{\phi}')^2 + \frac{\alpha}{\sqrt{\pi}}\widetilde{\phi}'\Box\widetilde{\phi}'_g \\ & - \frac{1}{2}\sum_{i_D}\left(\partial_\mu\widetilde{\phi}'_{i_D}\right)^2 - \frac{1}{2}(\partial_\mu\zeta')^2 + \frac{1}{2}\sum_{i_D}\left(\partial_\mu\zeta'_{i_D}\right)^2 - m'_o\sum_{j=1}^{N}\cos\left\{2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + 2\sqrt{\frac{\pi}{N}}\widetilde{\phi}'_g\right. \\ & \left. + \frac{2}{\sqrt{N}}\frac{g\sqrt{(N-1)}}{\sqrt{1-\frac{g^2(N-1)}{\pi}}}\widetilde{\phi}' + \sum_{i_D=1}^{N-1}\lambda_{jj}^{i_D}\left(2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}\right)\right\},\end{aligned} \tag{B.11}$$

where

$$\alpha^2 = \frac{g^2(N-1)}{1-\frac{g^2(N-1)}{\pi}}, \tag{B.12}$$

the fields $\widetilde{\phi}', \widetilde{\phi}'_{i_D}, \zeta'$ and ζ'_{i_D} are defined by (3.35)–(3.38) and

$$\widetilde{\phi}'_g = \sqrt{\frac{N}{\pi}}\widetilde{\phi}_g. \tag{B.13}$$

The field ζ_g is a pure gauge excitation and does not appear in the bosonized gauge invariant Lagrangian. Let us consider the following term in the Lagrangian (B.11):

$$\mathcal{L}(\widetilde{\phi}'_g, \widetilde{\phi}') = \frac{\pi}{2Ne^2}(\Box\widetilde{\phi}'_g)^2 - \frac{1}{2}(\partial_\mu\widetilde{\phi}'_g)^2 + \frac{1}{2}(\partial_\mu\widetilde{\phi}')^2 + \frac{\alpha}{\sqrt{\pi}}\widetilde{\phi}'\Box\widetilde{\phi}'_g. \tag{B.14}$$

The fields $\widetilde{\phi}'$ and $\widetilde{\phi}'_g$ can be decoupled by introducing the new field

$$\widetilde{\vartheta} = \widetilde{\phi}' - \frac{\alpha}{\sqrt{\pi}}\widetilde{\phi}'_g, \tag{B.15}$$

and we obtain

$$\mathcal{L}(\widetilde{\phi}'_g, \widetilde{\phi}') = \mathcal{L}(\widetilde{\vartheta}, \widetilde{\phi}''_g) = \frac{1}{2}(\partial_\mu\widetilde{\vartheta})^2 + \frac{1}{2m^2}(\Box\widetilde{\phi}''_g)^2 - \frac{1}{2}(\partial_\mu\widetilde{\phi}''_g)^2, \tag{B.16}$$

where we have defined

$$\widetilde{\phi}''_g = \left(1 - \frac{g^2(N-1)}{\pi}\right)^{-\frac{1}{2}}\widetilde{\phi}'_g, \tag{B.17}$$

and the mass parameter m is given by

$$m^2 = \frac{e^2N}{\pi}\left(1 - \frac{g^2(N-1)}{\pi}\right)^{-1}. \tag{B.18}$$

For $g^2 = 0$ we obtain the mass of the gauge field of QED_2 with N Fermi fields [19].

In order to 'dequartize' the Lagrangian for the field $\widetilde{\phi}''_g$ in (B.16), we shall consider the following functional integral identity,

$$\begin{aligned}
&\int \mathcal{D}\widetilde{\phi}''_g \mathrm{e}^{\mathrm{i}\int \mathrm{d}^2z\{\frac{1}{2m^2}(\Box\widetilde{\phi}''_g)^2+\frac{1}{2}\widetilde{\phi}''_g\Box\widetilde{\phi}''_g\}} \\
&\quad = \int \mathcal{D}\widetilde{\Xi}\,\mathcal{D}\widetilde{\phi}''_g\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^2z\{-\frac{1}{2}\widetilde{\Xi}^2+\frac{1}{m}(\Box\widetilde{\Xi})\widetilde{\phi}''_g+\frac{1}{2}\widetilde{\phi}''_g\Box\widetilde{\phi}''_g\}} \\
&\quad = \int \mathcal{D}\widetilde{\Sigma}\,\mathcal{D}\widetilde{\phi}''_g\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^2z\{-\frac{1}{2}m^2\widetilde{\Sigma}^2+(\Box\widetilde{\Sigma})\widetilde{\phi}''_g+\frac{1}{2}\widetilde{\phi}''_g\Box\widetilde{\phi}''_g\}} \\
&\quad = \int \mathcal{D}\widetilde{\Sigma}\,\mathcal{D}\widetilde{\eta}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^2z\{\frac{1}{2}\widetilde{\eta}\Box\widetilde{\eta}-\frac{1}{2}(\widetilde{\Sigma}\Box\widetilde{\Sigma}+m^2\widetilde{\Sigma}^2)\}},
\end{aligned} \tag{B.19}$$

where we used

$$\widetilde{\Sigma} = \frac{1}{m}\widetilde{\Xi}, \tag{B.20}$$

and the decoupling of the fields $\widetilde{\Sigma}$ and $\widetilde{\phi}''_g$ is performed by defining the new field

$$\widetilde{\eta} = \widetilde{\phi}''_g + \widetilde{\Sigma}. \tag{B.21}$$

The bosonized Lagrangian is then given by

$$\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\widetilde{\varphi})^2 + \frac{1}{2}(\partial_\mu\widetilde{\vartheta})^2 + \frac{1}{2}\sum_{i_D}(\partial_\mu\widetilde{\varphi}_{i_D})^2 - \frac{1}{2}\sum_{i_D}(\partial_\mu\widetilde{\phi}'_{i_D})^2 - \frac{1}{2}(\partial_\mu\zeta')^2 + \frac{1}{2}\sum_{i_D}\left(\partial_\mu\zeta'_{i_D}\right)^2 \\
& - \frac{1}{2}(\partial_\mu\widetilde{\eta})^2 + \frac{1}{2}(\partial_\mu\widetilde{\Sigma})^2 - \frac{1}{2}m^2\widetilde{\Sigma}^2 - m'_o\sum_{j=1}^{N}\cos\left\{2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + 2\frac{\alpha}{\sqrt{N}}\widetilde{\vartheta}\right. \\
& \left. + \frac{1}{\sqrt{N}}\sqrt{\frac{4\pi}{1-\frac{g^2(N-1)}{\pi}}}(\widetilde{\Sigma}+\widetilde{\eta}) + \sum_{i_D=1}^{N-1}\lambda^{i_D}_{jj}\left(2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}\right)\right\},
\end{aligned} \tag{B.22}$$

In order to obtain the bosonized theory in terms of the true bosonic degrees of freedom, let us introduce the canonical transformations

$$\beta\widetilde{\Phi} = 2\sqrt{\frac{\pi}{N}}\widetilde{\varphi} + 2\frac{\alpha}{\sqrt{N}}\widetilde{\vartheta}, \tag{B.23}$$

$$\beta\widetilde{\xi} = 2\frac{\alpha}{\sqrt{N}}\widetilde{\varphi} - 2\sqrt{\frac{\pi}{N}}\widetilde{\vartheta}, \tag{B.24}$$

with

$$\beta^2 = \frac{4\pi}{N}\frac{1}{1-\frac{g^2}{\pi}(N-1)}, \tag{B.25}$$

and

$$\gamma\widetilde{\Phi}_{i_D} = 2\sqrt{\pi}\widetilde{\varphi}_{i_D} + 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\phi}'_{i_D}, \tag{B.26}$$

$$\gamma\widetilde{\xi}_{i_D} = 2\frac{g}{\sqrt{1+\frac{g^2}{\pi}}}\widetilde{\varphi}_{i_D} + 2\sqrt{\pi}\widetilde{\phi}'_{i_D}, \tag{B.27}$$

with

$$\gamma^2 = \frac{4\pi}{1 + \frac{g^2}{\pi}}. \tag{B.28}$$

In this way we obtain from (B.22) the effective bosonized Lagrangian (5.6).

Appendix C. Perturbative expansion

To begin with, let us define

$$\Xi_j(z) \doteq \beta\Phi(z) + \gamma \sum_{i_D} \lambda_{jj}^{i_D} \Phi_{i_D}(z). \tag{C.1}$$

Following the standard procedure [13, 21], expanding the exponentials in (6.1) in powers of m_o' and using a multinomial expansion, we get

$$\begin{aligned}
\exp\left\{-m_o' \sum_{j=1}^{N} \int d^2z \cos \Xi_j(z)\right\} &= \sum_{n=0}^{\infty} \frac{(-m_o')^n}{n!} \left(\sum_{j=1}^{N} \int d^2z \cos \Xi_j(z)\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(-m_o')^n}{n!} \sum_{m_1, m_2, \ldots, m_N} \frac{(n!)\delta_{n, m_1+\cdots+m_N}}{m_1! m_2! \cdots m_N!} \prod_{j=1}^{N} \left(\int d^2z \cos \Xi_j(z)\right)^{m_j} \\
&= \sum_{n=0}^{\infty} \frac{(-m_o')^n}{n!} \sum_{m_1, m_2, \ldots, m_N} \frac{(n!)\delta_{n, m_1+\cdots+m_N}}{m_1! m_2! \cdots m_N!} \prod_{j=1}^{N} \left(\int \prod_{k_j=1}^{m_j} d^2z_{k_j} \cos \Xi_j(z_{k_j})\right),
\end{aligned} \tag{C.2}$$

where the summation $\sum_{m_1, m_2, \ldots, m_N}$ runs over all positive-integer values of m_j for which

$$\sum_{j=1}^{N} m_j = n. \tag{C.3}$$

In terms of the exponential of Ξ_j, the expansion (C.2) can be written as

$$\begin{aligned}
\exp\left\{-m_o' \sum_{j=1}^{N} \int d^2z \cos \Xi_j(z)\right\} &= \sum_{n=0}^{\infty} \frac{(-m_o')^n}{n!} \\
&\quad \times \sum_{m_1, m_2, \ldots, m_N} \frac{(n!)\delta_{n, m_1+\cdots+m_N}}{m_1! m_2! \cdots m_N!} \prod_{j=1}^{N} \left(\frac{1}{2^{m_j}} \sum_{\{\alpha_{k_j}\}_{m_j}} \int \prod_{k_j=1}^{m_j} d^2z_{k_j}\, e^{i \sum_{k_j=1}^{m_j} \alpha_{k_j} \Xi_j(z_{k_j})}\right),
\end{aligned} \tag{C.4}$$

where $\alpha_{k_j} = \pm 1$ and $\sum_{\{\alpha_{k_j}\}_{m_j}}$ runs over all possibilities in the set $\{\alpha_1, \ldots, \alpha_{m_j}\}$. The partition function is then given by

$$\begin{aligned}
\mathcal{Z} &= \frac{1}{\mathcal{Z}_o} \sum_{n=0}^{\infty} \frac{(-m_o')^n}{2^n (n!)} \sum_{m_1, m_2, \ldots, m_N} \frac{(n!)\delta_{n, m_1+\cdots+m_N}}{m_1! m_2! \cdots m_N!} \left(\prod_{j=1}^{N} \int \prod_{k_j=1}^{m_j} d^2z_{k_j} \sum_{\{\alpha_{k_j}\}_{m_j}}\right) \\
&\quad \times \int \mathcal{D}\Phi\, e^{-S^{(0)}(\Phi)}\, e^{\int d^2z J(z)\Phi(z)} \int \prod_{i_D=1}^{N-1} \mathcal{D}\Phi_{i_D}\, e^{-S_{i_D}^{(0)}(\Phi_{i_D})} e^{\int d^2z \sum_{i_D} J^{i_D}(z)\Phi_{i_D}(z)},
\end{aligned} \tag{C.5}$$

where

$$J(z) = \mathrm{i}\beta \sum_{j=1}^{N} \left(\sum_{k_j=1}^{m_j} \alpha_{k_j} \delta^{(2)}(z - z_{k_j}) \right), \tag{C.6}$$

$$J^{i_D}(z) = \mathrm{i}\gamma \sum_{j=1}^{N} \left(\sum_{k_j=1}^{m_j} \alpha_{k_j} \lambda^{i_D}_{jj} \delta^{(2)}(z - z_{k_j}) \right). \tag{C.7}$$

Performing the functional integration over the fields $\widetilde{\Phi}$ and $\widetilde{\Phi}_{i_D}$, we obtain

$$\begin{aligned} \mathcal{Z} = \frac{1}{\mathcal{Z}_o} \sum_{n=0}^{\infty} \frac{(-m_o')^n}{2^n} \sum_{m_1, m_2, \ldots, m_N} \frac{\delta_{n, m_1 + \cdots + m_N}}{m_1! m_2! \cdots m_N!} \left(\prod_{j=1}^{N} \int \prod_{k_j=1}^{m_j} \mathrm{d}^2 z_{k_j} \right) \\ \times \sum_{\{\alpha_{k_j}\}_{m_j}} \exp \left\{ -\frac{\beta^2}{2} \sum_{j=1}^{N} \sum_{j'=1}^{N} \sum_{k_j=1}^{m_j} \sum_{\bar{k}_{j'}=1}^{m_{j'}} \alpha_{k_j} \alpha_{\bar{k}_{j'}} D_o(z_{k_j} - z_{\bar{k}_{j'}}) \right\} \\ \times \exp \left\{ -\frac{\gamma^2}{2} \sum_{i_D=1}^{N-1} \sum_{j=1}^{N} \sum_{j'=1}^{N} \sum_{k_j=1}^{m_j} \sum_{\bar{k}_{j'}=1}^{m_{j'}} \alpha_{k_j} \alpha_{\bar{k}_{j'}} \lambda^{i_D}_{jj} \lambda^{i_D}_{j'j'} D_o(z_{k_j} - z_{\bar{k}_{j'}}) \right\}, \end{aligned} \tag{C.8}$$

where

$$D_o(z) = \lim_{\mu^2 \to 0} \Delta(z; \mu) = -\frac{1}{4\pi} \ln\{-\mu^2(|z|^2 + \varepsilon^2)\} \tag{C.9}$$

is the infrared and ultraviolet regularized massless Green's function of the two-dimensional Laplacian operator.

Appendix D. The model with $N = 2$

In this appendix we shall make some remarks related to the problems of regularization and the factorization of the partition function. In [3] the bosonic structure of the model with interaction among two ($N = 2$) different species of massive Fermi fields has been discussed within the functional integral framework. The bosonized partition function is obtained and the main conclusion given in [3] is that 'for a specific value of the coupling constant ($g^2 = 2\pi/3, \beta_- = 0$) one boson field (in our notation $\widetilde{\Phi}_-$) becomes a free field with negative metric (a ghost field), while the other ($\widetilde{\Phi}_+$) is a sine-Gordon field, and the model becomes equivalent to the ordinary sine-Gordon model of a single boson field'. The 'equivalence', as proposed in [3], was stablished between the partition functions and not between the generating functionals, and thus does not imply the isomorphism between the corresponding Hilbert spaces of states. Free massless scalar fields decouple in the partition function, but do not decouple in the generating functional. From our point of view, this 'equivalence' is a misleading conclusion, which is a consequence—first—of the factorization of the partition function and—second—of disregarding the free massless field ζ_j (the field χ_j in the notation of [3]), so it cannot be considered as being an intrinsic property of the model with $N = 2$. The use of a regularization prescription that breaks the local gauge invariance of the fermionic piece of the effective Lagrangian (3.9) leads to the appearance of an effective coupling constant depending on the regularization parameter. In this case, the model exhibits distinct physical and ghost ranges. In [13], the bosonic structure of the model with $N = 2$ was analyzed using an arbitrary regularization parameter $\boldsymbol{a}$, which is related to the regularization parameter $\boldsymbol{b}$ used in equation (3.17) by

$$\frac{\boldsymbol{a}}{2\pi} = \frac{1}{4\pi} - \frac{\boldsymbol{b}}{g^2}. \tag{D.1}$$

The model exhibits distinct ranges for the relation between the coupling constant g^2 and the regularization parameter $\boldsymbol{a}$. In [13] the model was considered for $0 \leqslant \boldsymbol{a} < 1$ and in the range

$$0 \leqslant g^2 < \frac{\pi}{(1-\boldsymbol{a})}, \qquad \frac{g^2\boldsymbol{a}}{\pi} < 1. \tag{D.2}$$

This is a physical range, which includes the free theory as a limit, the model does not exhibit sine-Gordon ghost fields, the two physical Bose fields $\widetilde{\Phi}_\pm$ have positive metric quantization and the unitarity is not spoiled. The value $\boldsymbol{a} = 0\,(\boldsymbol{b} = g^2/4\pi)$ corresponds to a regularization that preserves the local gauge invariance of the fermionic piece of the effective Lagrangian (3.9).[7]

In order to discuss the conclusion of [3] and to show that the procedure of making $\beta = 0$ is meaningless for a sine-Gordon theory, let us consider the bosonization of the model with $N = 2$ and a regularization parameter $\boldsymbol{a} > 1$. In this case the bosonized Lagrangian corresponding to (3.33) is given by ($\widetilde{\phi} \equiv \widetilde{\phi}_+, \widetilde{\phi}_{i_D} \equiv \widetilde{\phi}_-$, etc)

$$\begin{aligned}\mathcal{L}_{\text{eff}} = {} & \frac{1}{2}(\partial_\mu\widetilde{\varphi}_+)^2 + \frac{1}{2}(\partial_\mu\widetilde{\varphi}_-)^2 + \frac{1}{2g^2}\left(1 + \frac{g^2}{\pi}(\boldsymbol{a}-1)\right)(\partial_\mu\widetilde{\phi}_+)^2 \\ & - \frac{1}{2g^2}\left(1 - \frac{g^2}{\pi}(\boldsymbol{a}-1)\right)(\partial_\mu\widetilde{\phi}_-)^2 - \frac{1}{2}\frac{1}{g^2}\left(1 - \frac{g^2}{\pi}\boldsymbol{a}\right)(\partial_\mu\zeta_+)^2 \\ & + \frac{1}{2g^2}\left(1 + \frac{g^2}{\pi}\boldsymbol{a}\right)(\partial_\mu\zeta_-)^2 - m_o'\cos(\sqrt{2\pi}\widetilde{\varphi}_+ + \sqrt{2}\widetilde{\phi}_+)\cos(\sqrt{2\pi}\widetilde{\varphi}_- + \sqrt{2}\widetilde{\phi}_-).\end{aligned} \tag{D.3}$$

Note that the range for the values of the 'effective' coupling constant $g^2(\boldsymbol{a}-1) \lesseqgtr \pi$ in (D.3) determines the sign of the metric for the field $\widetilde{\phi}_-$. As in the standard Thirring model, we shall consider the range for which

$$g^2 < \frac{\pi}{(\boldsymbol{a}-1)}, \qquad \frac{g^2\boldsymbol{a}}{\pi} < 1. \tag{D.4}$$

The condiction (D.4) ensures that in this range the metrics for the fields $\widetilde{\phi}_-$ and ζ_+ in (D.3) remain fixed. Performing the field scalings and the canonical transformations, similar to (3.40)–(3.44), the sine-Gordon parameters $\beta_\pm$ are given by

$$\beta_\pm^2 = \frac{2\pi^2 \pm 2\pi g^2\boldsymbol{a}}{\pi \pm g^2(\boldsymbol{a}-1)}. \tag{D.5}$$

The sine-Gordon parameters given in [3] are obtained from (D.5) by making $g^2 \to g^2/2$ and with the regularization parameter $\boldsymbol{a} = 3$. The value $g^2\boldsymbol{a} = \pi$, for which in a naive way $\beta_- = 0$ and the field $\widetilde{\phi}_-$ becomes a 'would be' free field (as proposed in [3] at the partition function level), violates condition (D.4) and the field ζ_+ in (D.3) is not a dynamical degree of freedom. In other words, if one starts at the beginning by considering $g^2\boldsymbol{a} = \pi$, the Bose field algebra collapses and the fermion–boson mapping becomes meaningless. In this case, the free massless fields ζ_- and ξ_+ give contributions to the fermionic Wightman functions. The mass operator is now given by

$$m_o'\cos\beta_+\widetilde{\Phi}_+\cos 2\sqrt{\pi}(\widetilde{\varphi}_- + \widetilde{\phi}'_-). \tag{D.6}$$

[7] In this case the ghost range is for $g^2\boldsymbol{a} > \pi$ and $g^2 > \pi/(1-\boldsymbol{a})$ [13].

Due to the opposite metric quantization for the fields $\widetilde{\varphi}_-$ and $\widetilde{\phi}'_-$, the field combination $(\widetilde{\varphi}_- + \widetilde{\phi}'_-)$ is a zero norm field and does not contribute to the fermionic Wightman functions. The resulting Hilbert space contains contributions from the redundant free massless Bose field algebra, which are not present in the original Fermionic model. This also can be easily checked in the model with massless fermions for which the exact Wightman functions can be computed[8].

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[8] The same problem occurs in the bosonization of the standard massive Thirring model using an arbitrary regularization parameter [9]. The choice of a gauge noninvariant regularization implies a redefinition of the β parameter of the sine-Gordon theory and thus the physical range for the coupling constant. Using an arbitrary regularization parameter $\boldsymbol{a} > 1$, as defined by equation (D.1), the bosonized Lagrangian is given by

$$\mathcal{L} = \frac{1}{2g^2}\left(1 - \frac{\boldsymbol{a}g^2}{\pi}\right)(\partial_\mu \zeta)^2 + \frac{1}{2}(\partial_\mu \widetilde{\varphi})^2 - \frac{1}{2g^2}\left(1 - \frac{g^2}{\pi}(\boldsymbol{a} - 1)\right)(\partial_\mu \widetilde{\phi})^2 - m_o' \cos\left(2\sqrt{\pi}\widetilde{\varphi} + 2\widetilde{\phi}\right).$$

For $g^2\boldsymbol{a} < \pi$, and $g^2(\boldsymbol{a} - 1) < \pi$, the sine-Gordon parameter is given by

$$\beta^2 = \frac{4\pi(\pi - g^2\boldsymbol{a})}{\pi - g^2(1 - \boldsymbol{a})}.$$

The same wrong conclusion can arise by considering the value $g^2\boldsymbol{a} = \pi$ that violates condition (D.2) and for which $\beta = 0$. Starting at the beginning with $g^2\boldsymbol{a} = \pi$, the field ζ is not a dynamical degree of freedom and the Lagrangian is given by $(\widetilde{\phi} = \sqrt{\pi}\widetilde{\phi}')$,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \widetilde{\varphi})^2 - \tfrac{1}{2}(\partial_\mu \widetilde{\phi})^2 - m_o' \cos 2\sqrt{\pi}(\widetilde{\varphi} + \widetilde{\phi}').$$

Due to the opposite metric for the fields $\widetilde{\varphi}$ and $\widetilde{\phi}'$, in this case the scale dimension of the mass operator is $D = 0$ and the Fermion–Boson mapping collapses.

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